Minimal Viscosity Solution of HJB equation and Optimal Consumption and Investment with Tax

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Outline

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   - Approximating Problem
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This a joint work with X.Chen and M.Dai

Equation

- Equation is the following variational inequality

\[ \mathcal{F}[u] = \max \{ \mathcal{L}u - \frac{1}{q}(u_x)^q, \mathcal{B}u, \mathcal{I}u \} = 0, (x, y, k) \in S \quad (1) \]

- Here differential operators

\[ \mathcal{L}u = \frac{1}{2} \sigma^2 y^2 u_{yy} + \mu y u_y + rx u_x - \beta u \]
\[ \mathcal{B}u = -u_x + u_y + u_k, \quad \mathcal{I}u = [(1 - \alpha)y + \alpha k]u_x - y u_y - ku_k \]

- State space

\[ S = \{(x, y, k) \mid y > 0, k > 0, z = x + (1 - \alpha)y + \alpha k > 0\} \]

- Assume \( 0 < p < 1, \ q = \frac{p}{p-1} < 0; \ 0 < \alpha < 1; \ \sigma, r, \mu, \beta > 0 \)
Definition: state constrained viscosity solutions

If $V(x, y, k)$ is continuous on $\bar{S}$ and satisfies the following equations in viscosity sense

$$-\mathcal{F}[V] = 0 \text{ in } S, \quad -\mathcal{F}[V] \leq 0 \text{ on } \Gamma_2$$

Then $V(x, y, k)$ is called a state constrained viscosity solution.

$V(x, y, k)$ is a subsolution on $S \cup \Gamma_2$, a supersolution in $S$. 
Other boundary condition

- State constrained boundary condition

\[ \Gamma_2 = \{(x, y, k) \in \partial S \mid z > 0\} \]

- Dirichlet boundary condition

\[ \Gamma_1 = \{(x, y, k) \in \partial S \mid z = 0\}, \]

\[ V = 0 \text{ on } \Gamma_1 \quad (3) \]

- Boundary condition at \( \infty \)

\[ V \in \mathcal{C} := \left\{ u \in C(\bar{S}) \mid \sup_{(x, y, k) \in S} \frac{|u(x, y, k)|}{|x + (1 - \alpha)y + \alpha k|^p} < \infty \right\} \quad (4) \]
Non-uniqueness

Assumption

\[ \beta > \beta_p := p \left[ r + \frac{(\mu - r)^2}{2\sigma^2[1 - p]} \right] \]

Non-uniqueness

- \( \hat{V} = Az^p \) is a classic solution of (2) on \( S \cup \Gamma_2 \) provided \( A \gg 1 \)
- \( \hat{V} = Az^p \) is a viscosity solution of problem (2)-(4) provided \( A \gg 1 \)
Non-uniqueness

Remark

$A^{1-q} \geq \frac{p^q}{q(\beta_p - \beta)}$, $\beta_p = p \left[ r + \frac{(\mu-r)^2}{2(1-p)\sigma^2} \right]$

Classic solution via viscosity solution

Counter example

$G[u] = \max\{\mathcal{L}u + U^*(u_x), -Bu\}$

Comparison principle fails for problem (2)-(4)
Background

- **Optimal Consumption and Investment with Tax**
  - Continuous-time investment and consumption with capital gains tax
  - Singular stochastic control problem

- **Bellman DPP: Bellman equation**
  - Value function is a viscosity solution of related Bellman equation
  - Comparison principle holds for viscosity solution of Bellman equation
  - Value function is a unique viscosity solution of related Bellman equation
  - Find solution of Bellman equation: analytically or numerically.

- **Comparison principle** Implies verification theorem and convergence for numeric scheme.
Questions

- **Question 1: Nonuniqueness**
  - In fact, Bellman equation admits many viscosity solutions. How to decide which solution is value function?
  - It is necessary to find a criterion to select the right viscosity solution for the value function.

- **Question 2: The convergence for numeric scheme**
  - Comparison principle fails
  - It is a singular stochastic control problem
P.L. Lions: The limitation of a sequence of viscosity solutions

I. Ben Tahar, H. M. Soner, and N. Touzi


Financial Model

- $B_t$-Bond(Cash) $dB_t = rB_t dt, \ r > 0$;
- $S_t$-risky asset
  \[ dS_t = S_t[\mu dt + \sigma dW_t], \quad t \geq 0 \]
- $x_t$-Cash amount
- $y_t \geq 0$-amount invested in the stock according to current price $S_t$
- $k_t \geq 0$-amount invested in the stock according to basis(purchase) price
Control in Financial Model

- $L_t - dL_t$: the dollar amount transferred from the cash to the stock account at time $t$
- $M_t - dM_t \leq 1$: the proportion of shares transferred from the stock account to the cash at time $t$
- $C_t \geq 0$: consumption rate
State equations

- The discounted wealth process invested in cash (tax rate $-\alpha$)

$$dx_t = (rx_t - C_t)dt - dL_t + [y_t - \alpha(y_t - k_t)]dM_t, \quad (5)$$

- The discounted wealth process invested in the stock

$$dy_t = \mu y_t dt + \sigma y_t dW_t + dL_t - y_t dM_t \quad (6)$$

- Purchase price process

$$dk_t = dL_t - k_t dM_t \quad (7)$$

- The solvency condition

$$z_t = x_t + y_t - \alpha(y_t - k_t) = x_t + (1 - \alpha)y_t + \alpha k_t \geq 0$$
Control problem and value function

- State space
  \[ z = x + (1 - \alpha)y + \alpha k, \quad S = \{(x, y, k) \mid y > 0, k > 0, z > 0\} \]

- \(A\)-the set of admissible control
  \[ A(x, y, k) = \{(C, L, M) \mid dL_t \geq 0, \ 0 \leq dM_t \leq 1, \ (x_t, y_t, k_t) \in \bar{S}\} \]

- Value function \((x, y, k) \in S\)
  \[ V(x, y, k) = \sup_{(L,M,C) \in A} E\left[ \int_0^\tau e^{-\beta t}U(C_t)dt \mid x_0 = x, k_0 = y, k_0 = k \right] \]
  \[ \text{(8)} \]
  subject to state equation (5)-(7)

- Stopping time
  \[ \tau := \sup\{t > 0 \mid z_s > 0 \ \forall \ s \in [0, t)\} \]
Notations

- Utility $U(c) = \frac{1}{p} c^p$
- Dual utility
  \[ U^*(d) = \sup_{c \geq 0} \{ U(c) - cd \} = -\frac{1}{q} d^q, \quad \frac{1}{q} + \frac{1}{p} = 1 \]
- Equation
  \[ \mathcal{F}[u] = \max \{ \mathcal{L}u + U^*(u_x), \mathcal{B}u, \mathcal{I}u \} \]
Bellman equation and boundary conditions

**Theorem**

Assume

\[ \beta > \beta_p = p \left[ r + \frac{(\mu - r)^2}{2(1 - p)\sigma^2} \right] \]  \hspace{1cm} (9)

Then the value function \( V(x, y, k) \) in (8) is a continuous state constrained viscosity solution of the following problem

\[ -\mathcal{F}[V] = 0 \text{ in } S, \quad V = 0 \text{ on } \Gamma_1, \quad -\mathcal{F}[V] \leq 0 \text{ on } \Gamma_2 \]  \hspace{1cm} (10)

**Boundary condition at \( \infty \)**

\[ V \in \mathcal{C} := \left\{ u \in C(\bar{S}) \left| \sup_{(x,y,k) \in S} \frac{|u(x, y, k)|}{z^p} < \infty \right. \right\} \]
Non-uniqueness

Theorem

Non-uniqueness

(1) \( \hat{V} = Az^p \) is a classic solution of problem (10) provided \( A \gg 1 \)

(2) \( \hat{V} = Az^p \) is a viscosity solution of problem (10) provided \( A \gg 1 \)

Provided

\[ A^{1-q} \geq \frac{p^q}{q(\beta_p - \beta)} \]
Approximating control problem and value function

- **Admissible set**

\[ A_\lambda(x, y, k) = \{(C, L, M) \in \mathcal{A} \mid dL_t \leq \lambda z_t dt, \quad dM_t \leq \lambda dt \} \]

- **Value function**

\[ V_\lambda(x, y, k) = \sup_{(L, M, C) \in A_\lambda} E\left[ \int_0^\tau \frac{1}{p} e^{-\beta t} C_t^p dt \mid x_0 = x, k_0 = y, k_0 = k \right] \]

subject to (5)-(7)
Theorem

Assume (9). Then the value function $V_\lambda(x, y, k)$ in (11) is a continuous viscosity solution of the following problem

$$-\mathcal{F}_\lambda[V] = 0 \quad \text{in} \quad S, \quad -\mathcal{F}_\lambda[V] \leq 0 \quad \text{on} \quad \Gamma_2, \quad V = 0 \quad \text{on} \quad \Gamma_1 \quad (12)$$

Furthermore, we have

$$V_\lambda \in \mathcal{C} := \left\{ u \in C(\overline{S}) \mid \sup_{(x,y,k) \in S} \frac{|u(x, y, k)|}{|x + (1 - \alpha)y + \alpha k|^p} < \infty \right\}$$

Operator

$$\mathcal{F}_\lambda[u] = \mathcal{L}u + U^*(u_x) + \lambda z(\mathcal{B}u)^+ + \lambda (\mathcal{I}u)^+$$
Theorem

Comparison principle

Assume (9). If $u \in \mathcal{C}$ is a viscosity subsolution and $v$ is a viscosity supersolution in $\mathcal{C}$, and $u \leq v$ on $\Gamma_1$. Then $u \leq v$ on $S$. Consequently, (12) admits at most one viscosity solution in $\mathcal{C}$.

Idea in proof:

- Compare to $\mathcal{F}_\lambda[u]$ with $\mathcal{F}[u]

\[ \mathcal{F}_\lambda[u] = Lu + U^*(u_x) + \lambda z(Bu)^+ + \lambda(Su)^+ \]

- $\mathcal{F}_\lambda(M,p,u,x,y,k)$ is increase in $\lambda$
Comparison principle

- Estimate

\[
0 \leq \left\{ \frac{\sigma^2}{2}(y_1^2m_1^{22} - y_2^2m_2^{22}) + \mu y_1 \Phi_{y_1}(\xi_i, \eta_i) + \mu_2 y_2 \Phi_{y_2} + rx_1 \Phi_{x_1} + rx_2 \Phi_{x_2} \right\}
+ \left\{ U^* \left( g_{x_1}(\xi_i) + \Phi_{x_1}(\xi_i, \eta_i) \right) - U^* \left( -g_{x_1}(\eta_i) - \Phi_{ix_2}(\xi_i, \eta_i) \right) \right\}
+ \left\{ \mathcal{L}^1 g(\xi_i) + \mathcal{L}^1 g(\eta_i) - \beta[u(\xi_i) - v(\eta_i) - g(\xi_i) - g(\eta_i)] \right\} + \left\{ \lambda z_2[\mathcal{S}^2 \phi(\xi_i, \eta_i)] + \lambda z_1[\mathcal{S}^2 \phi(\xi_i, \eta_i)]^+ \right\}
+ \left\{ \lambda[\mathcal{B}^1 \phi(\xi_i, \eta_i)]^+ - \lambda[-\mathcal{B}^2 \phi(\xi_i, \eta_i)]^+ \right\}
\]
Comparison principle

- **Estimate**

\[
\lambda z_1 [\mathcal{B}^1 \phi(\xi_i, \eta_i)]^+ - \lambda z_2 [-\mathcal{B}^2 \phi(\xi_i, \eta_i)]^+ \leq \lambda z_1 [\mathcal{B}^1 g(\xi_i)]^+
\]

\[
+ \lambda z_2 [\mathcal{B}^2 g(\eta_i)]^+ + \lambda z_2 [\mathcal{B}^1 \Phi(\xi_i, \eta_i) + \mathcal{B}^2 \Phi(\xi_i, \eta_i)]^+ + \lambda (z_1 - z_2) [\mathcal{B}^1 \Phi(\xi_i, \eta_i)]
\]

\[
\leq 2\lambda q [g(\xi_i) + g(\eta_i)] + 0 + \lambda |\xi_i - \eta_i| [2i^2 |\xi_i - \eta_i| + 2i] (1 + b)
\]

\[
\leq 2\lambda q [g(\xi_i) + g(\eta_i)] + 4\lambda (1 + b)^2.
\]
Comparison principle

- **Estimate**

\[
\lambda[\mathcal{S}^1 \phi]^+ - \lambda[-\mathcal{S}^2 \phi_2]^+ \leq \lambda[\mathcal{S}^1 \phi + \mathcal{S}^2 \phi]^+ \\
\leq \lambda[2k^2|\xi_i - \eta_i|^2 + 2k|(\xi_i - \eta_i)|] \\
\leq - \left(\beta_q - 2\lambda q\right)\left[g(\xi_i) + g(\eta_i)\right] - \left(\beta a - [\sigma^2 + 4\mu + 2r + 4(2 + b)\lambda + \beta]^2\right)
\]

Hence, first taking \(q \in (p, 1)\) such that \(\beta_q > 0\) and then taking sufficiently small we then obtain a contradiction.
Singular Control Approximation

**Theorem**

Singular Control Approximation

\[
\lim_{\lambda \to \infty} V_\lambda(x, y, k) = V(x, y, k), \quad \forall (x, y, k) \in D
\]

- Idea in proof:
  - \( V_\lambda(x, y, k) \) is increase in \( \lambda \)
  - \( \mathcal{F}_\lambda(M, p, u, x, y, k) \) is increase in \( \lambda \)
  - \( \mathcal{F}_\lambda(M, p, u, x, y, k) \to \mathcal{F}(M, p, u, x, y, k) \) as \( \lambda \to \infty \)
  - Compare to P.L.Lions’s work
Singular Control Approximation

\[ z_t = z_{t}^{p,C} - e^{rt} \left\{ b\eta_{t}^{L} + (1 - \alpha)\eta_{t}^{\pi} + \alpha\eta_{t}^{k} \right\} \]

where

\[ z_{t}^{p,C} = xe^{rt} + (1 - \alpha)yS_{t} + \alpha k - \int_{0}^{t} e^{(t-s)r} c_{s} ds \]

\[ \eta_{t}^{L} = \int_{0}^{t} e^{-rs} dL_{s} \]

\[ \eta_{t}^{\pi} = \int_{0}^{t} \left[ S_{s}e^{-rs} - S_{t}e^{-rt} \right] d\pi_{s} \]

\[ \eta_{t}^{k} = \int_{0}^{t} \left[ e^{-rs} - e^{-rt} \right] dk_{s} \]
Theorem

**Minimal Viscosity Solution Selection Principle**

Assume (9). Then $V(x, y, k)$ is the minimal viscosity solution of (10) in $C$.

- Ideas in Proof
  - By comparison principle of approximating problem (12)
Conclusions and Remarks

- **Conclusions**
  - We prove the comparison principle of approximating problem (12)
  - We prove the value function of approximating problem tends the value function of primal problem
  - We prove the value function of primal problem is the minimal viscosity solution of (10)
  - We establish a framework for numerical simulation

- **Remarks**
  - Compare to Soner’s work, Lions’s works
  - Compare to transaction cost problem

- **Open Problem**
  - A class of PDE
Thank You!