On Radial solutions to the $A_2$ and $B_2$ Chern-Simons systems

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In the last few decades, various Chern-Simons field theories have been widely studied, largely motivated by their applications to the physics of high critical temperature superconductivity.

The physical models in these Chern-Simons theories can be reduced to systems of nonlinear partial differential equations, which have posed many mathematically challenging problems.


\[ \Delta u_a + \sum_{b=1}^{r} K_{ab} e^{u_b} - \sum_{b=1}^{r} \sum_{c=1}^{r} e^{u_b} K_{ab} e^{u_c} K_{bc} = 4\pi \sum_{j=1}^{N_a} \delta_{p_j^a}, \quad 1 \leq a \leq r, \]  

(1)

where \( K = (K_{ab}) \) is a \( r \times r \) matrix with \( \det K > 0 \), \( \delta_p \) denotes the Dirac measure at \( p \) in \( \mathbb{R}^2 \). See (Yang, CMP1997) for the derivation of (1) from the non-Abelian Chern-Simons field theory.

For any \( 1 \leq a \leq r \), \( u_a : \mathbb{R}^2 \setminus \{p_j^a \mid 1 \leq j \leq N_a\} \rightarrow \mathbb{R} \) satisfies

\[ \Delta u_a + \sum_{b=1}^{r} K_{ab} e^{u_b} - \sum_{b=1}^{r} \sum_{c=1}^{r} e^{u_b} K_{ab} e^{u_c} K_{bc} = 0 \]

with prescribed asymptotic behaviors

\[ u_a(x) = 2n_{p_j^a} \ln |x - p_j^a| + O(1) \quad \text{near } p_j^a. \]
Let \((K^{-1})_{ab}\) denote the inverse of the matrix \(K\). We always assume

\[
\sum_{b=1}^{r}(K^{-1})_{ab} > 0, \quad a = 1, 2, \ldots, r. \tag{2}
\]

A solution \(\mathbf{u} = (u_1, \cdots, u_r)\) of (1) is called a **topological solution** if

\[
u_a(x) \to \ln \left( \sum_{b=1}^{r}(K^{-1})_{ab} \right) \quad \text{as} \ |x| \to +\infty, \quad a = 1, \cdots, r,
\]

a solution \(\mathbf{u}\) is called a **non-topological solution** if

\[
u_a(x) \to -\infty \quad \text{as} \ |x| \to +\infty, \quad a = 1, \cdots, r.
\]
The existence of topological solutions of system (1) was completely solved by (Yang, CMP1997) via variational methods.

However, the existence of non-topological solutions is much more difficult and remains open for a long time.

To simplify the problem, in the sequel we only consider $2 \times 2$ case, i.e.

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} + K \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} - K \begin{pmatrix} e^{u_1} \\ 0 \end{pmatrix} K \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} = \begin{pmatrix} 4\pi \sum_{j=1}^{N_1} \delta p_{j}^1 \\ 4\pi \sum_{j=1}^{N_2} \delta p_{j}^2 \end{pmatrix}, \quad (3)$$

where $K = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. 
When $K$ is one of the following three types of Cartan matrix of rank 2:

$$
\mathbf{A}_2 (\text{i.e. } SU(3)) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},
$$

(Ao, Lin and Wei (Mem. AMS2013, CAG2014) proved the existence of non-topological solutions for the first time. Their idea is to apply the powerful finite-dimensional reduction method. Therefore, it is worthy to study these three Cartan matrices separately.

Their proof indicates that, this system poses different difficulties for different choices of the matrix $K$. Therefore, it is worthy to study these three Cartan matrices separately.
On the other hand, the understanding of the structure of non-topological solutions is far from complete. We will see later that this system should have infinitely many non-topological solutions.

Again to simplify the problem, from now on, we focus on the radially symmetric solutions of (3). So all the vortices coincide at the origin. Then system (3) turns to be

\[
\begin{pmatrix}
\Delta u_1 \\
\Delta u_2
\end{pmatrix} + K \begin{pmatrix}
e^{u_1} \\
e^{u_2}
\end{pmatrix} - K \begin{pmatrix}
e^{u_1} & 0 \\
0 & e^{u_2}
\end{pmatrix} K \begin{pmatrix}
e^{u_1} \\
e^{u_2}
\end{pmatrix} = \begin{pmatrix}
4\pi N_1 \delta_0 \\
4\pi N_2 \delta_0
\end{pmatrix}
\]

in \( \mathbb{R}^2 \), \( \text{(5)} \)

where \( N_1, N_2 \in \mathbb{N} \cup \{0\} \).
For $K = A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, i.e. the $A_2$ Chern-Simons system:

$$\begin{cases}
\Delta u_1 + 2e^{u_1} - e^{u_2} - 4e^{2u_1} + e^{u_1+u_2} + 2e^{2u_2} = 4\pi N_1 \delta_0 \\
\Delta u_2 + 2e^{u_2} - e^{u_1} - 4e^{2u_2} + e^{u_1+u_2} + 2e^{2u_1} = 4\pi N_2 \delta_0
\end{cases} \text{ in } \mathbb{R}^2 \tag{6}$$

A solution $(u_1, u_2)$ of (6) is a **topological solution** if

$$(u_1, u_2) \to (0, 0) \quad \text{as} \quad |x| \to +\infty;$$

a **non-topological solution** if

$$(u_1, u_2) \to (-\infty, -\infty) \quad \text{as} \quad |x| \to +\infty;$$

a **mixed-type solution** if $(u_1, u_2) \to (\ln \frac{1}{2}, -\infty)$ or $(u_1, u_2) \to (-\infty, \ln \frac{1}{2})$ as $|x| \to +\infty$. 

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Define a continuous function $J : \mathbb{R}^2 \to \mathbb{R}$ by

$$J(x, y) = x^2 + xy + y^2.$$  

(7)

**Theorem A.** (Huang-Lin, CMP2013) Suppose that $(u_1, u_2) \neq (0, 0)$ is a radially symmetric solution of system (6). Then both $u_1 < 0$ and $u_2 < 0$ in $\mathbb{R}^2$, and one of the following conclusions holds.

(i) $(u_1, u_2)$ is a topological solution.

(ii) $(u_1, u_2)$ is a mixed-type solution.

(iii) $(u_1, u_2)$ is a non-topological solution and there exist constants $\alpha_1, \alpha_2 > 1$ such that

$$u_j(x) = -2\alpha_j \ln |x| + O(1) \text{ as } |x| \to +\infty, \quad j = 1, 2.$$ 

(8)

Consequently, $e^{u_1}, e^{u_2} \in L^1(\mathbb{R}^2)$. Moreover, $(\alpha_1, \alpha_2)$ satisfies

$$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1).$$ 

(9)
Remark that the inequality (9) follows from the Pohozaev identity:

$$J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1) = \frac{3}{2} \int_0^\infty r \left[ e^{2u_1} + e^{2u_2} - e^{u_1+u_2} \right] dr.$$ 

Therefore, (9) is a necessary condition for the existence of radial non-topological solutions satisfying the asymptotic condition (8).

**Question:** Given $\alpha_1, \alpha_2 > 1$ satisfying (9):

$$J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1) > 0.$$ 

Is there a radial non-topological solution of system (6) subject to the asymptotic condition (8)?

$$u_j(x) = -2\alpha_j \ln |x| + O(1) \text{ as } |x| \to +\infty, \quad j = 1, 2.$$
If we let \( N_1 = N_2 = N \) and \( u_1 = u_2 = u \) in (6), then system (6) turns to be the following Chern-Simons-Higgs equation

\[
\Delta u + e^u(1 - e^u) = 4\pi N \delta_0 \quad \text{in} \quad \mathbb{R}^2.
\] (10)

Equation (10) is known as the \( SU(2) \) Chern-Simons equation for the Abelian case. Let \( u \) be a radial non-topological solution of (10) and satisfies \( u(x) = -2\alpha \ln |x| + O(1) \) near \( \infty \). Then the Pohozaev identity gives

\[
(\alpha - 1)^2 - (N + 1)^2 = \frac{1}{2} \int_0^\infty r e^{2u} \, dr > 0,
\]

which implies \( \alpha > N + 2 \). (Chan,Fu,Lin CMP2002) proved that the inequality \( \alpha > N + 2 \) is also a sufficient condition for the existence of radial non-topological solutions \( u \) with \( u(x) = -2\alpha \ln |x| + O(1) \) near \( \infty \).
However, this might not hold for system (6) with the asymptotic condition (8). The reason is following: Suppose \((u_{1,n}, u_{2,n})\) are any sequence of non-topological solutions such that

\[
\int_{0}^{\infty} r \left[ e^{2u_{1,n}} + e^{2u_{2,n}} - e^{u_{1,n}+u_{2,n}} \right] \, dr \to 0.
\]

Then the corresponding \((\alpha_{1,n} - 1, \alpha_{2,n} - 1)\) might converge to \((N_1 + 1, N_2 + 1)\) only.

Therefore, roughly speaking, the inequality (9):

\[
J(\alpha_{1} - 1, \alpha_{2} - 1) - J(N_{1} + 1, N_{2} + 1) > 0
\]

should not be a sufficient condition for the existence of radial non-topological solutions satisfying (8).
A sufficient condition for the $A_2$ case

**Theorem B.** (Choe-Kim-Lin,CMP2014) *Let $N_1$, $N_2$ be non-negative integers. Define*

$$S = \left\{ (\alpha_1, \alpha_2) \mid \begin{align*}
-2N_1 - N_2 - 3 &< \alpha_2 - \alpha_1 < N_1 + 2N_2 + 3 \\
2\alpha_1 + \alpha_2 &> N_1 + 2N_2 + 6 \\
\alpha_1 + 2\alpha_2 &> 2N_1 + N_2 + 6
\end{align*} \right\}.$$

*Then for each fixed $(\alpha_1, \alpha_2) \in S$, the $A_2$ Chern-Simons system*

$$\begin{cases} 
\Delta u_1 + 2e^{u_1} - e^{u_2} - 4e^{2u_1} + e^{u_1+u_2} + 2e^{2u_2} = 4\pi N_1 \delta_0 & \text{in } \mathbb{R}^2 \\
\Delta u_2 + 2e^{u_2} - e^{u_1} - 4e^{2u_2} + e^{u_1+u_2} + 2e^{2u_1} = 4\pi N_2 \delta_0
\end{cases}$$

*has a radial non-topological solution $(u_1, u_2)$ subject to the asymptotic condition*

$$u_j(x) = -2\alpha_j \ln |x| + O(1) \text{ as } |x| \to +\infty, \quad j = 1, 2.$$
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Their idea: the Larey-Schauder degree theory

Given \((\alpha_1, \alpha_2) \in S\), deform the \(A_2\) Chern-Simons system for \(t \in [0, 1]\):

\[
\begin{align*}
\Delta u_1 + 2e^{u_1} - e^{u_2} - 4e^{2u_1} + e^{u_1+u_2} + 2e^{2u_2} &= 4\pi t N_1 \delta_0 \\
\Delta u_2 + 2e^{u_2} - e^{u_1} - 4e^{2u_2} + e^{u_1+u_2} + 2e^{2u_1} &= 4\pi t N_2 \delta_0
\end{align*}
\text{in } \mathbb{R}^2
\]

\[u_j(x) = -2\alpha_j(t) \ln |x| + O(1) \text{ as } |x| \to +\infty, \quad j = 1, 2.\]

where \((\alpha_1(1), \alpha_2(1)) = (\alpha_1, \alpha_2)\) and \(\alpha_1(0) = \alpha_2(0) > 2\). Then establish a priori estimates. Two advantages:

(1). To obtain the a priori estimates, the limiting equation of blowup analysis is either a single equation or the \(SU(3)\) Toda system:

\[
\begin{align*}
\Delta u_1 + 2e^{u_1} - e^{u_2} &= 4\pi t N_1 \delta_0 \\
\Delta u_2 + 2e^{u_2} - e^{u_1} &= 4\pi t N_2 \delta_0
\end{align*}
\text{in } \mathbb{R}^2, \quad \int e^{u_j} dx < \infty.
\]

Solutions of \(SU(n+1)\) Toda system are completely classified by (Lin-Wei-Ye, Invent. Math 2012).

(2). Symmetry for \(t = 0\): \((u_1, u_2) \leftrightarrow (u_2, u_1)\). So it suffices to consider the distribution of \(u_1 = u_2\) when computing the Leray-Schauder degree for \(t = 0\).
Recall $\mathbf{B}_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$.

**Theorem 1**

Let $N_1, N_2$ be non-negative integers. Then for any $(\alpha_1, \alpha_2)$ satisfying

$$\alpha_1 > N_1 + N_2 + 3 \quad \text{and} \quad \alpha_2 > 2N_1 + N_2 + 4,$$

the $\mathbf{B}_2$ Chern-Simons system

$$\begin{cases} 
\Delta u_1 + 2e^{u_1} - 2e^{u_2} - 4e^{2u_1} + 2e^{2u_2} = 4\pi N_1 \delta_0 \\
\Delta u_2 + 2e^{u_2} - 2e^{u_1} - 4e^{2u_2} + 2e^{u_1+u_2} + 4e^{2u_1} = 4\pi N_2 \delta_0
\end{cases} \quad \text{in } \mathbb{R}^2$$

has a radial non-topological solution $(u_1, u_2)$ subject to the asymptotic condition

$$u_j(x) = -2\alpha_j \ln |x| + O(1) \quad \text{as} \quad |x| \to +\infty, \quad j = 1, 2.$$
We prove Theorem 1 also by applying the degree theory. Difference: No symmetry \((u_1, u_2) \rightarrow (u_2, u_1)\) for \(B_2\) system, so it is no helpful to compute the degree by deforming \(tN_j\).

Our deform: Given \((\alpha_1, \alpha_2)\) satisfying (11), we deform for \(t \in [0, 1]\):

\[
\begin{aligned}
\begin{cases}
(\Delta u_1) + K(t) \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} - K(t) \begin{pmatrix} e^{u_1} & 0 \\ 0 & e^{u_2} \end{pmatrix} K(t) \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} = \left( \frac{4\pi N_1}{4\pi N_2} \delta_0 \right) \quad \text{in } \mathbb{R}^2, \\
u_j(x) = -2\alpha_j(t) \ln |x| + O(1) \quad \text{as } |x| \to +\infty, \quad j = 1, 2.
\end{cases}
\end{aligned}
\]

where \(K(1) = B_2\) and \(K(0) = A_2\); \((\alpha_1(1), \alpha_2(1)) = (\alpha_1, \alpha_2)\) and \((\alpha_1(0), \alpha_2(0)) \in S\).

**Advantage:** the degree for \(t = 0\) is known nonzero by Theorem B.
Theorem 2 (A priori estimates)

There exists a constant $C > 0$ independent of $t \in [0, 1]$ such that

$$\| u_1 - f_1 \|_{L^\infty(\mathbb{R}^2)} + \| u_2 - f_2 \|_{L^\infty(\mathbb{R}^2)} \leq C, \quad \forall \ t \in [0, 1].$$  (13)

Here

$$f_k(x) = f_k(x; t) := 2N_k \ln |x| - (\alpha_k(t) + N_k) \ln(1 + |x|^2), \quad k = 1, 2.$$  

Disadvantage: By a blowup analysis, one limiting equation is

$$\left( \Delta u_1 \Delta u_2 \right) + K(t) \left( e^{u_1} e^{u_2} \right) = \left( \frac{4\pi N_1}{4\pi N_2} \delta_0 \right) \times \delta_0 \quad \text{in} \ \mathbb{R}^2, \quad \int e^{u_j} < \infty,$$

solutions of which can be known only for $t = 0$ and $t = 1$. This poses new difficulties.

By Theorem 2, the Larey-Schauder degree is well-defined under some functional space for $t \in [0, 1]$ and independent of $t$. 
Remark: The Cartan matrix $K = G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ case remains open, because the above approach of degree theory can not apply. This provides another evidence that this Chern-Simons system with different matrix $K$ might be essentially different and need to treat separately via different ideas.

A natural question: *Is the range given in Theorem 1 and 2 an optimal range of $(\alpha_1, \alpha_2)$ for the existence of radial solutions satisfying*

$$u_j(x) = -2\alpha_j \ln |x| + O(1) \text{ as } |x| \to +\infty, \quad j = 1, 2.$$
The answer is NO for the $B_2$ case.

**Theorem 3**

Let $(\alpha_1, \alpha_2)$ satisfy either

\[ \alpha_1 = N_1 + N_2 + 3, \quad \alpha_2 > 1 \quad \text{or} \quad \alpha_2 = 2N_1 + N_2 + 4, \quad \alpha_1 > 1. \]

Then the $B_2$ Chern-Simons system admits a sequence of radial non-topological bubbling solutions $(u_{1,n}, u_{2,n})$ such that $\sup_{\mathbb{R}^2} u_{2,n} \to -\infty$ (or $\sup_{\mathbb{R}^2} u_{1,n} \to -\infty$) as $n \to \infty$ and there exist constants $\alpha_{1,n} > 1, \, \alpha_{2,n} > 1$ such that

\[ u_{k,n}(r) = -2\alpha_{k,n} \ln r + O(1) \quad \text{as} \quad r \to \infty, \quad k = 1, 2, \]

and $(\alpha_{1,n}, \alpha_{2,n}) \to (\alpha_1, \alpha_2)$ as $n \to \infty$. 

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Remark: \( \sup_{\mathbb{R}^2} u_{2,n} \to -\infty \) as \( n \to \infty \) implies

\[
\| u_{2,n} - f_{2,n} \|_{L^\infty(\mathbb{R}^2)} \to +\infty
\]

where \( f_{2,n}(x) := 2N_2 \ln |x| - (\alpha_{2,n} + N_2) \ln(1 + |x|^2) \). Therefore, the a priori estimates (i.e. Theorem 2) can not hold for

\[
\alpha_1 = N_1 + N_2 + 3, \quad \alpha_2 > 1 \quad \text{and} \quad \alpha_2 = 2N_1 + N_2 + 4, \quad \alpha_1 > 1.
\]

In conclusion, the range given in Theorem 1:

\[
\alpha_1 > N_1 + N_2 + 3 \quad \text{and} \quad \alpha_2 > 2N_1 + N_2 + 4
\]

is an optimal range of \((\alpha_1, \alpha_2)\) for the existence of radial solutions satisfying

\[
u_j(x) = -2\alpha_j \ln |x| + O(1) \quad \text{as} \quad |x| \to +\infty, \quad j = 1, 2
\]

in view of the degree theory, but not the optimal range for the existence of non-topological solutions.
Main idea: the shooting method

\[
\begin{aligned}
\Delta u_1 + 2e^{u_1} - 4e^{2u_1} - e^{u_2} + 2e^{2u_2} &= 4\pi N_1 \delta_0 \\
\Delta u_2 + 2e^{u_2} - 2e^{u_1} - 4e^{2u_2} + 2e^{u_1 + u_2} + 4e^{2u_1} &= 4\pi N_2 \delta_0
\end{aligned}
\text{ in } \mathbb{R}^2
\]

Inspired by the proof the a priori estimates, we define

\[ \gamma := \alpha_1 + \alpha_2 + 1 > N_1 + 2. \]

Then by (Chan-Fu-Lin, CMP2002), there is a unique radial solution \( U \) of the Chern-Simons-Higgs equation

\[
\begin{aligned}
\Delta U + 2e^U - 4e^{2U} &= 4\pi N_1 \delta_0 \text{ in } \mathbb{R}^2, \\
U(x) &= -2\gamma \ln |x| + O(1) \text{ as } |x| \to \infty.
\end{aligned}
\]

Let \( V(|x|) = V(x) := U(x) - 2N_1 \ln |x|, \) then \( V(0) := \lim_{r \to 0} V(r) \) is well defined.
To use the shooting method, we consider an initial problem of system (6) in a radial variable:

\[
\begin{aligned}
&u_1''(r) + \frac{1}{r} u_1'(r) = -(2e^{u_1} - e^{u_2} - 4e^{2u_1} + 2e^{2u_2}), \quad r > 0, \\
u_2''(r) + \frac{1}{r} u_2'(r) = -(2e^{u_2} - 2e^{u_1} - 4e^{2u_2} + 2e^{u_1+u_2} + 4e^{2u_1}), \quad r > 0, \\
u_1(r) = 2N_1 \ln r + V(0) + o(1), \quad r \to 0, \\
u_2(r) = 2N_2 \ln r + \ln \epsilon + o(1), \quad r \to 0,
\end{aligned}
\]

where \( \epsilon \in (0, 1) \). Clearly, the solution of (15) depends on \( \epsilon \) and we denote it by \((u_1, \epsilon, u_2, \epsilon)\).

Since the solution \((u_1, \epsilon, u_2, \epsilon)\) of the initial problem (15) exists locally, the key point is to prove that \((u_1, \epsilon, u_2, \epsilon)\) exists globally for \( r \in (0, +\infty) \) (i.e. does not blow up at finite \( r \)) provided that \( \epsilon > 0 \) is sufficiently small. This is the most difficult part of the proof.
To overcome this difficulty, we need to give a delicate analysis of the asymptotic behavior of \((u_{1,\varepsilon}, u_{2,\varepsilon})\). For example, we need to understand what happens if \(u_{1,\varepsilon}\) and \(u_{2,\varepsilon}\) intersect and how many times they intersect (exactly two times!). The main tool is the well-known Pohozaev identity together with the blow up analysis.

**A interesting phenomena:** There exist two intersection points \(R_{3,\varepsilon} \gg R_{1,\varepsilon} \gg 1\) of \(u_{1,\varepsilon}\) and \(u_{2,\varepsilon}\) such that:

1. \(u_{1,\varepsilon} \to U\) in \(C_{loc}^2(B(0, R_{1,\varepsilon}))\) as \(\varepsilon \to 0\).
2. \(\int_{R_{1,\varepsilon}}^{R_{3,\varepsilon}} r e^{u_{1,\varepsilon}} \, dr \to 0\), \(\int_0^{R_{1,\varepsilon}} r e^{u_{2,\varepsilon}} \, dr \to 0\), \(\int_{R_{3,\varepsilon}}^{\infty} r e^{u_{2,\varepsilon}} \, dr \to 0\), and \(\int_{R_{1,\varepsilon}}^{R_{3,\varepsilon}} r e^{u_{2,\varepsilon}} \, dr \to \gamma + N_1 + N_2 + 1\), \(\int_{R_{3,\varepsilon}}^{\infty} r e^{u_{1,\varepsilon}} \, dr \to \frac{4}{3} (\alpha_1 - 1)\).

Namely, \(u_{1,\varepsilon}, u_{2,\varepsilon}\) blow up in different regions at infinity at the same time.
Recall the mixed-type solution: $(u_1, u_2) \to (\ln \frac{1}{2}, -\infty)$ or $(u_1, u_2) \to (-\infty, \ln \frac{1}{2})$ as $|x| \to +\infty$.

**Theorem C.** (Choe-Kim-Lin, CVPDE2017) Let $N_1, N_2$ be non-negative integers. Then for any $\alpha > N_1 + 2N_2 + 3$, the $A_2$ Chern-Simons system has a radial mixed-type solution $(u_1, u_2)$ satisfying

$$u_1(x) \to \ln \frac{1}{2}, \quad u_2(x) = -2\alpha \ln |x| + O(1) \quad \text{as} \quad |x| \to \infty.$$ 

They used the variational method. The range $\alpha > 2N_1 + N_2 + 3$ should not be sharp.
Mixed-type solutions with delicate characterization

Theorem 4

Let $N_1 = N_2 = 0$. Then for any $\alpha > 2$, the $A_2$ Chern-Simons system has a radial mixed-type solution $(u_1, u_2) = (u_1^\alpha, u_2^\alpha)$ satisfying

1. $u_1(r) \to \ln \frac{1}{2}$, $u_2(r) = -2\alpha \ln r + O(1)$ as $r \to \infty$;

2. $u_1(r) > u_2(r)$ and $u_1(r) \geq \ln \frac{9 + 4\sqrt{3}}{33}$ for all $r \geq 0$;

3. $u_2'(r) < 0$ and $(u_1 + u_2)'(r) < 0$ for all $r > 0$;

4. There exists an open subset $C \subsetneq (2, \infty)$ such that, if $\alpha \in C$, then there exists $r_\alpha > 0$ such that $(u_1^\alpha)'(r) < 0$ for $r \in (0, r_\alpha)$ and $(u_1^\alpha)'(r) > 0$ for $r > r_\alpha$; if $\alpha \notin C$, then $(u_1^\alpha)'(r) > 0$ for $r > 0$;

5. If $\alpha \to \infty$, then the initial value $(u_1^\alpha(0), u_2^\alpha(0)) \to (0, 0)$; If $\alpha \to 2$, then $(u_1^\alpha(0), u_2^\alpha(0)) \to (\ln \frac{1}{2}, -\infty)$.
**Remark:** If $N_j > 0$ for some $j$, then $ru'_1(r) + ru'_2(r) \to 2N_1 + 2N_2$ as $r \to 0$, which implies that properties (3)-(4) can not hold. Therefore, $N_1 = N_2 = 0$ is a necessary condition to obtain such kind of mixed-type solutions as stated in Theorem 4.

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A difficult question: Whether any mixed-type solution for the case $N_1 = N_2 = 0$ satisfy these properties or not? We still have no answer to this question so far, but we believe so, for example

**Lemma 5**

Let $(u_1, u_2) \neq (0, 0)$ be either non-topological or mixed-type radial solution.

- If $N_1 = N_2 = 0$, then $(u_1 + u_2)'(r) < 0$ for all $r > 0$.
- If $(N_1, N_2) \neq (0, 0)$, then $(u_1 + u_2)'(r)$ changes sign exactly once.

$$ru_1'(r) + ru_2'(r) \to 2N_1 + 2N_2 \text{ as } r \to 0.$$
Consider an initial value problem in a radial variable:

\[
\begin{align*}
\begin{cases}
    u_1''(r) + \frac{1}{r} u_1'(r) &= -(2e^{u_1} - e^{u_2} - 4e^{2u_1} + e^{u_1+u_2} + 2e^{2u_2}), & r > 0, \\
    u_2''(r) + \frac{1}{r} u_2'(r) &= -(2e^{u_2} - e^{u_1} - 4e^{2u_2} + e^{u_1+u_2} + 2e^{2u_1}), & r > 0, \\
    u_1(0) &= a, & u_2(0) &= b, & u_1'(0) &= u_2'(0) &= 0.
\end{cases}
\end{align*}
\]

(15)

Roughly speaking, given any \( b < 0 \), since the initial value \((0, b)\) gives a finite-time blowup solution, we define

\[
g(b) := \inf \{ a_0 < 0 \mid \text{the initial value } (a, b) \text{ gives a finite-time blowup solution for any } a \in [a_0, 0] \}
\]

Then as long as \( g(b) > -\infty \), the initial value \((g(b), b)\) will give an entire solution \((u_1^b, u_2^b)\).
First, we can prove that $g(b) > b$ for all $b < 0$ and hence the entire solution $(u_1^b, u_2^b)$ is well-defined.

Since $(0, 0)$ is the unique radial topological solution, so $(u_1^b, u_2^b)$ is either non-topological or mixed-type.

On the other hand, it can be proved that

$$\{(a, b) \mid (a, b) \text{ gives a non-topological solution}\}$$

is open.

Therefore, this solution $(u_1^b, u_2^b)$ must be mixed-type.

In conclusion, the initial value $(g(b), b)$ gives a mixed-type solution for any $b < 0$.

The difficult part is to prove such mixed-type solutions satisfy the properties in Theorem 4. Our proof is based on delicate ODE analysis.
We can prove

\[ g(b) \geq \ln \frac{9 + 4\sqrt{3}}{33} \quad \text{for all } b < 0; \quad g(b) \nearrow \ln \frac{1}{2} \text{ as } b \to -\infty. \]

Formally, the picture of \( g(b) \) might be as follows (a conjecture about the initial value \((a, b)\)).
The property
\[ u_1(r) \geq \ln \frac{9 + 4\sqrt{3}}{33} \quad \text{for all} \quad r \geq 0 \]
is an consequence of \( u_1(r) > u_2(r) \) for all \( r \): Suppose \( u_1(r_0) = \inf u_1 \), then the first equation
\[ 2e^{u_1} - e^{u_2} - 4e^{2u_1} + e^{u_1+u_2} + 2e^{2u_2} \leq 0 \quad \text{at} \quad r_0. \]
Then
\[ \Delta = 33e^{2u_1} - 18e^{u_1} + 1 \geq 0 \quad \text{at} \quad r_0, \]
gives
\[ e^{u_1(r_0)} \geq \frac{9 + 4\sqrt{3}}{33} \quad \text{or} \quad e^{u_1(r_0)} \leq \frac{9 - 4\sqrt{3}}{33} \]
The second possibility can be removed by \( u_1(r_0) > u_2(r_0) \).
Thank you!

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On Radial solutions to the $A_2$ and $B_2$ Chern-Simons systems