Finite time blow up for a reaction-diffusion system in bounded domain *

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Abstract. This paper mainly considers the coupled parabolic system in a bounded domain: $u_t = \Delta u + u^\alpha v^p$, $v_t = \Delta v + u^q v^\beta$ in $\Omega \times (0, T)$ with null Dirichlet boundary value condition which had been discussed by Wang in [15]. The aim of this paper is to solve the open problem mentioned in the Remark of [15].

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1 Introduction

In this paper, we consider heat system

$$\begin{cases} u_t = \Delta u + u^\alpha v^p, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + u^q v^\beta, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\alpha, \beta, p, q \geq 0$, $\alpha + p > 0, \beta + q > 0$, initial data $0 \leq u_0, v_0 \in C(\bar{\Omega})$ satisfy the compatibility conditions.

This system had been studied by M.X. Wang in [15], for related work we refer readers to [1, 4, 5, 6, 9, 10, 11, 13]. More results on the related studies were collected in the surveys [3, 8] and the books [7, 14]. Before recalling the conclusion of [15], we give some notation first. Let $\lambda_1 > 0$ be the first eigenvalue to

$$-\Delta\varphi = \lambda\varphi \ \text{ in } \Omega, \quad \varphi = 0 \ \text{ on } \in \partial\Omega, \tag{1.2}$$

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and φ the first eigenfunction, normalized by $\varphi > 0$ in Ω, $\int_\Omega \varphi \, dx = 1$.

The main result in [15] is about blow-up and global existence of the solution of (1.1). As mentioned in [15] (see the Remark in that paper), the case $2/3 \le \lambda_1 < 1$ with some critical assumption was left open. We claim in that case, any nontrivial solutions must blow up in finite time, that is,

Theorem 1. *Assume that*

$$p > 0;\ q = 0;\ \beta = 1;\ \lambda_1 < 1,\ \alpha = 1 + p(1 - \lambda_1)/\lambda_1 > 1 \tag{1.3}$$

or

$$q > 0;\ p = 0;\ \alpha = 1;\ \lambda_1 < 1,\ \beta = 1 + q(1 - \lambda_1)/\lambda_1 > 1 \tag{1.4}$$

Then, for any initial data $u_0(x) \ge 0, \not\equiv 0; v_0 \ge 0, \not\equiv 0$; the solution of (1.1) blows up in finite time.

2 Proof of Theorem 1

In this section, we will prove the Theorem 1. The method we used here mainly comes from [2] (a modification of the method in [4] and [12]). Without loss of generality, we may (and will) suppose (1.3) is valid, then the problem (1.1) becomes,

$$\begin{cases} u_t = \Delta u + u^\alpha v^p, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + v, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

where $\alpha = 1 + p(1 - \lambda_1)/\lambda_1$.

At first show a lemma as preliminaries.

Lemma 2.1. ([2]) *Suppose that $S(t)$ is the semigroup generated by Laplacian with null boundary condition, and $u_0 \in L^1(\Omega)$ a nonnegative nontrivial function. Then for any $T_1 > 0$, there exists $c = c(T_1, u_0) > 0$ such that*

$$S(t)u_0 \ge c\mathrm{e}^{-\lambda_1 t}\varphi(x), \quad t \ge T_1. \tag{2.2}$$

Proof. For any $T_1 > 0$, by Hopf's Lemma, there exists $c = c(T_1, u_0) > 0$ such that

$$S(t)u_0 \ge c\varphi(x),\ (x,t) \in \Omega \times \{t = T_1\}.$$

Hence, $S(t)u_0$ is a supersolution of the following problem

$$\begin{cases} u_t = \Delta u, & (x,t) \in \Omega \times (T_1, \infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (T_1, \infty), \\ u(x,T_1) = c\varphi(x), & x \in \Omega. \end{cases} \tag{2.3}$$

On the other hand, $c\mathrm{e}^{-\lambda_1 t}\varphi$ is the unique solution of (2.3). This concludes

$$S(t)u_0 \geq c\mathrm{e}^{-\lambda_1 t}\varphi(x), \quad t \geq T_1,$$

by the comparison principle. $\square$

We now obtain our Theorem.

Proof of Theorem 1. Assume for contradiction that (u, v) is global.

We can obtain by (2.1) that

$$u(t) = S(t)u_0 + \int_0^t S(t-s)[u^\alpha(s)v^p(s)]ds, \tag{2.4}$$

Since $v_0 \not\equiv 0$, by Lemma 2.1 we can assume without loss of generality that

$$v(x,t) = e^t S(t)v_0 \geq ce^{(1-\lambda_1)t}\varphi(x). \tag{2.5}$$

Using Lemma 2.1 again, we have

$$u(x,t) \geq S(t)u_0(x) \geq c\mathrm{e}^{-\lambda_1 t}\varphi(x),\ t \geq T_1,$$

and rewrite as

$$u(x,t) \geq C_0 t^{\gamma_0}\mathrm{e}^{-\lambda_1 t}\varphi(x),\ t \geq t_0,$$

with $C_0 := c > 0$, $\gamma_0 := 0$, and $t_0 := T_1$.

We will prove that for $k = 0, 1, 2, \dots$ there exist an increasing sequence $\{t_k\}$, and sequences $\{\gamma_k\}$ and $\{C_k\}$, such that

$$u(x,t) \geq C_k t^{\gamma_k}\mathrm{e}^{-\lambda_1 t}\varphi(x), \quad t \geq t_k. \tag{2.6}$$

It suffices to prove (2.6) holds for the case of $k+1$, provided it is true for k.

Substitute (2.5) and (2.6) to (2.4). By Lemma 2.1, we have

$$\begin{aligned}
u(x,t) &\geq \int_{t_k}^t S(t-s)\{[C_k s^{\gamma_k}\mathrm{e}^{-\lambda_1 s}\varphi(x)]^\alpha[ce^{(1-\lambda_1)s}\varphi]^p\}ds \\
&\geq c^p C_k^\alpha \left(\int_{t_k}^t s^{\alpha\gamma_k}e^{-\lambda_1 s}S(t-s)\varphi^{1+\frac{p}{\lambda_1}}ds\right) \\
&\geq c^p C_k^\alpha \left(\int_{t_k}^t s^{\alpha\gamma_k}c_1(T_1,\varphi^{1+\frac{p}{\lambda_1}})e^{-\lambda_1 t}\varphi(x)ds\right) \\
&\geq \frac{2MC_k^\alpha}{\alpha\gamma_k+1}\left(t^{\alpha\gamma_k+1} - t_k^{\alpha\gamma_k+1}\right)e^{-\lambda_1 t}\varphi(x) \\
&\geq \frac{MC_k^\alpha}{\alpha\gamma_k+1}t^{\alpha\gamma_k+1}e^{-\lambda_1 t}\varphi(x), \quad t \geq 2^{\frac{1}{\alpha\gamma_k+1}}t_k, \\
&:= C_{k+1}t^{\gamma_{k+1}}\mathrm{e}^{-\lambda_1 t}\varphi(x), \quad t \geq t_{k+1}
\end{aligned} \tag{2.7}$$

where

$$M = \frac{c^p}{2}c_1(T_1,\varphi^{1+\frac{p}{\lambda_1}}),$$

$$\gamma_{k+1} = \alpha\gamma_k + 1,\ C_{k+1} = \frac{MC_k^\alpha}{\gamma_{k+1}},\ t_{k+1} = 2^{\frac{1}{\gamma_{k+1}}} t_k.$$

A simple calculation gives $\gamma_{k+1} = \frac{\alpha^{k+1}-1}{\alpha-1}$ for $k \geq -1$. We have

$$\ln C_{k+1} = \alpha \ln C_k + \ln M - \ln(\gamma_{k+1}),$$

and hence

$$\begin{aligned}
\frac{1}{\alpha^{k+1}} \ln C_{k+1} &\geq \frac{1}{\alpha^k} \ln C_k + \frac{1}{\alpha^{k+1}} \ln M - \frac{q+1}{\alpha^{k+1}} \ln \gamma_{k+1} \\
&\geq \ln C_0 + \sum_{m=0}^{k} \left[\frac{1}{\alpha^{m+1}} \ln M - \frac{q+1}{\alpha^{m+1}} \ln \gamma_{m+1} \right] \\
&\geq L
\end{aligned}$$

with $L = -|\ln C_0| - \sum_{m=0}^{\infty} \left[\frac{1}{\alpha^{m+1}} |\ln M| + \frac{q+1}{\alpha^{m+1}} |\ln \gamma_{m+1}| \right] > -\infty$.
As a result,

$$C_{k+1}^{\frac{1}{\gamma_{k+1}}} = [C_k^{\frac{1}{\alpha^{k+1}}}]^{\frac{\alpha^{k+1}}{\gamma^{k+1}}} = [C_k^{\frac{1}{\alpha^{k+1}}}]^{\frac{(\alpha-1)\alpha^{k+1}}{\alpha^{k+1}-1}} \geq \mathrm{e}^{L(\alpha-1)}.$$

Denote $T_0 = \sup_{k\geq 1} t_k < +\infty$ by the definition of t_k. We conclude

$$u(x,t) \geq C_{k+1} t^{\alpha_{k+1}} \mathrm{e}^{-\lambda_1 t} \varphi(x) \geq (\mathrm{e}^{L(\alpha-1)} t)^{\alpha_{k+1}} \mathrm{e}^{-\lambda_1 t} \varphi(x), \quad t \geq T_0.$$

A contradiction occurs by setting $t = T_0 + \mathrm{e}^{-L(\alpha-1)}$ and letting $k \to +\infty$.

The proof is complete. □

Remark 2.1. We mainly deal with the open case of [15] in this paper. It is mentioned that, for problem (1.1), all the blow-up cases could be proved by the method used in this paper.

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