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ASYMPTOTIC BEHAVIOR OF DIRICHLET EIGENVALUES ON A BODY COATED BY FUNCTIONALLY GRADED MATERIAL

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Abstract. We consider the physical problem of protecting a thermally conducting body from overheating by thermal barrier coatings on a bounded domain, which has two components with a thin coating surrounding the body (of metallic nature), subject to the Dirichlet boundary condition. The coating is composed of two layers, the pure ceramic part and the mixed part. The latter is assumed to be functionally graded material (FGM) that is meant to make a smooth transition from being metallic to being ceramic. The thermal tensor A is isotropic on the body, and anisotropic on the coating; and the size of thermal tensor may differ significantly in these components. Eigenfunction expansion of the interior temperature function indicates that small eigenvalues of the elliptic operator $u \mapsto -\nabla \cdot (A\nabla u)$ are desirable for the insulation of the body. Therefore, we are motivated to study the asymptotic behavior of the eigenpairs of the Dirichelt eigenvalue problem, as the thickness of the coating shrinks. Our results greatly generalize those by Rosencrans and Wang [8] where the case of single coating layer is considered. In particular, we find new optimal scaling relationship between the thickness of the coating and its thermal conductivity that guarantees at least the principal eigenvalue is small for any general FGMs.

1. **Introduction.** The physical problem of protecting a thermally conducting body from over-heating by thermal barrier coatings arises naturally in many situations, such as space crafts and turbine engine blades protected by thermal insulators. See Figure 1, where Ω_1 represents the isotropically conducting body (say, of metallic nature) surrounded by a thin coating Ω_2 (say, of ceramic nature) with uniform thickness $\delta > 0$, i.e., $\Omega_2 = \{x \notin \Omega_1 \,|\, 0 < \mathrm{dist}(x, \partial\Omega_1) < \delta\}$. The (ceramic) coating Ω_2 protects the body Ω_1 from high temperature experienced during operation.

Let $A(x) = (a_{ij}(x))$ be a symmetric and positive-definite matrix describing the thermal tensor of the conducting medium with $x \in \Omega := \overline{\Omega_1} \cup \Omega_2$. We say that the heat flow is *isotropic* (the same in all directions) if A is invariant under all coordinate rotations, which is equivalent to saying that A is a multiple of the identity matrix I. Otherwise we call the heat conduction *anisotropic*. Currently, anisotropic thermal conductivity is a common macroscopic feature of nanocomposite materials used

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