Global existence and finite time blow-up of solutions of a Gierer-Meinhardt system *

Fang Li,[a†] Rui Peng[b‡] and Xianfa Song[c§]

a. Center for Partial Differential Equations, East China Normal University, 500 Dong Chuan Road, Minhang, 200241, Shanghai, China.

b. School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, Jiangsu Province, China.

c. Department of Mathematics, School of Science, Tianjin University, Tianjin, 300072, China.

July 14, 2016

Abstract

We are concerned with the Gierer-Meinhardt system with zero Neumann boundary condition:

$$\begin{cases} u_t = d_1\Delta u - a_1 u + \frac{u^p}{v^q} + \delta_1(x), & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v - a_2 v + \frac{u^r}{v^s} + \delta_2(x), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $p > 1$, $s > -1$, q, r, d_1, d_2, a_1, a_2 are positive constants, δ_1, δ_2, u_0, v_0 are nonnegative smooth functions, $\Omega \subset \mathbb{R}^d\,(d \geq 1)$ is a bounded smooth

*F. Li was partially supported by NSF of China (No. 11201148), Shanghai Pujiang Program (No. 13PJ1402400), and R. Peng was partially supported by NSF of China (No. 11271167, 11571200), Natural Science Fund for Distinguished Young Scholars of Jiangsu Province (No. BK20130002), and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

†E-mail: fangli0214@gmail.com

‡E-mail: pengrui_seu@163.com

§E-mail: songxianfa@tju.edu.cn

domain. We obtain new sufficient conditions for global existence and finite time blow-up of solutions, especially in the critical exponent cases: $p-1=r$ and $qr=(p-1)(s+1)$.

Keywords: Gierer-Meinhardt system; Solution; Global existence; Finite time blow-up.

1 Introduction

In this paper, of our concern is the following general Gierer-Meinhardt system

$$\begin{cases} u_t = d_1\Delta u - a_1 u + \frac{u^p}{v^q} + \delta_1(x), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - a_2 v + \frac{u^r}{v^s} + \delta_2(x), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $p>1$, $s>-1$, q, r, d_1, d_2, a_1, a_2 are positive constants, δ_1, δ_2, u_0, v_0 are nonnegative smooth functions, $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and the space dimension $d\geq 1$, and ν is the outward norm vector.

Following the idea of *diffusion-driven instability* proposed by Turing [41], Gierer and Meinhardt [12] in 1972 introduced the reaction-diffusion system (1.1) to model the pattern formation of spatial tissue structures of hydra in morphogenesis, a biological phenomenon which was discovered by Trembley in 1744 [42]. It is noted that in the original Gierer-Meinhardt model, δ_1 is a nonnegative constant and $\delta_2\equiv 0$; the general Gierer-Meinhardt model (1.1) was proposed in [14].

The Gierer-Meinhardt system (1.1) is one of the most famous models in biological pattern formation and belongs to the activator-inhibitor type. In the past few decades, the Gierer-Meinhardt system (1.1) has received extensive attention in research. The existence and uniqueness of a local solution to (1.1) is a folklore fact of standard parabolic theory; see, for example, [27], for details. Throughout the paper, a solution of (1.1) always means a classical nonnegative one. From the pure mathematical point of view, a fundamental question is the global existence of solution to (1.1). By a global solution of (1.1) we mean that its maximum existence time $T_{max}=\infty$, and by a blow-up solution (u,v) we mean that its maximum existence time $T_{max}<\infty$ and $\lim_{t\to T_{max}^-}\sup_{x\in\Omega}(u(x,t)+v(x,t))=\infty$. In the paper, unless otherwise stated, we assume that the initial data (u_0,v_0) satisfy

$$u_0(x)\geq 0,\quad v_0(x)>0,\quad \forall x\in\overline{\Omega}.$$

According to the strong maximum principle for parabolic equations, the solution