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Existence of standing waves of nonlinear Schrödinger equations with potentials vanishing at infinity

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ABSTRACT

For a singularly perturbed nonlinear elliptic equation $\varepsilon^2 \Delta u - V(x)u + u^p = 0$, $x \in \mathbb{R}^N$, we prove the existence of bump solutions concentrating around positive critical points of V when nonnegative V is not identically zero for $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ or nonnegative V satisfies $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 \log|x| > 0$ for $p = \frac{N}{N-2}$.

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1. Introduction and statement of main result

We consider the following nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \Delta \psi - V(x)\psi + |\psi|^{p-1}\psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.1)$$

A solution of the form $\psi(t, x) \equiv \exp(-iEt/\hbar)u(x)$ is a standing wave, where the function $u(x)$ satisfies the following equation

$$\frac{\hbar^2}{2} \Delta u - (V(x) - E)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N. \quad (1.2)$$

Eq. (1.2) arises in many field of physics, for instance, when we describe the propagation of light in some nonlinear optical materials and describe the interaction with each group of identical particles in ultra-states as Bose–Einstein condensates (see the introduction and references in [10]).

Without loss of generality, we can write $V(x) - E$ as $V(x)$, i.e., we shift E to 0. We are concerned with the existence of semiclassical standing waves, that is, solutions of (1.2) for small \hbar . Thus we study the following equation

$$\varepsilon^2 \Delta u - V(x)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} u(x) = 0 \quad (1.3)$$

when $\varepsilon > 0$ is sufficiently small.

In their pioneering work, Floer and Weinstein [17], applying the Lyapunov–Schmidt reduction method, constructed one bump solution of (1.3) concentrating around any nondegenerate critical point of $V(x)$ when $N = 1$, $p = 3$ and $0 < \inf_{x \in \mathbb{R}^N} V(x) < \sup_{x \in \mathbb{R}^N} V(x) < \infty$. There are many further works following their approach (see [2,4,13,19,22,24], and the references therein). On the other hand, for more general class of nonlinearities, a variational approach to the problem was suggested by Rabinowitz in [25]. This approach was developed further in [7,8,14–16,26–29]. In all the referred works, the condition $\inf_{x \in \mathbb{R}^N} V(x) > 0$ is essential.

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Byeon and Wang [10,11] gave the first breakthrough for this condition. When $\min_{\mathbb{R}^N} V(x) = 0$, $\liminf_{|x| \rightarrow \infty} V(x) > 0$, using a variational technique, they found locally minimal energy solutions of (1.3) concentrating around each isolated components of the zero set of V . On the contrary, the study of the case, where $V > 0$ but $\inf_{x \in \mathbb{R}^N} V(x) = 0$, has been initiated by Ambrosetti, Felli and Malchiodi [3]. From the work [3], there are further works developed the condition of V . Ambrosetti, Malchiodi and Ruiz [5] showed the existence of solutions concentrating around isolated stable critical points of V when $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 > 0$. In [30], Yin and Zhang found solutions concentrating around local minimum when dimension $N \geq 5$, $p \in (1, \frac{N+2}{N-2})$ and $V(x)$ is nonnegative. Moreover, Moroz and Van Schaftingen [23] improved this result under $N \geq 3$ and $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$. Bae and Byeon [6] extended these results for the general nonlinearity $f(u)$ satisfying Berestycki–Lions condition. Moreover, they exhibit a threshold of the asymptotic behavior of V at infinity between existence and nonexistence of solutions. When $N \geq 3$ and $\lim_{t \rightarrow 0} f(t)/t^\mu = 0$ for $\mu > \frac{N}{N-2}$, there exist solutions concentrating on local minimum points of nonnegative V . For $\varepsilon = 1$, $N \geq 3$ and $f(u) = u^p$, there exists no solution when $p \in (1, \frac{N}{N-2})$ and $\limsup_{|x| \rightarrow \infty} V(x)|x|^2 < \frac{2}{2-(p-1)(N-2)}$ or $p = \frac{N}{N-2}$ and $\limsup_{|x| \rightarrow \infty} V(x)|x|^2 \log|x| < \frac{(N-2)^2}{2}$.

In this paper, using Lyapunov–Schmidt reduction method, we will construct a solution concentrating around some positive critical point of nonnegative V not only local minimum points; which can be compared the work [6,23,30]. For $N \geq 3$ and $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$, we will allow that nonnegative V is not identically zero, i.e. V may have a compact support. And we will allow that nonnegative V satisfies $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 \log|x| > 0$ when dimension $N \geq 3$ and $p = \frac{N}{N-2}$.

We denote the set of positive critical points of $V(x)$ by $D = \{x \in \mathbb{R}^N \mid \nabla V(x) = 0, V(x) > 0\}$. For any set $A \subset \mathbb{R}^N$ and $\alpha > 0$, we define $A^\alpha \equiv \{x \in \mathbb{R}^N \mid \text{dist}(x, A) \leq \alpha\}$.

We assume the following conditions on V and D .

- (V1) For a small constant $\delta > 0$, $V(x) \in C(\mathbb{R}^N; \mathbb{R}) \cap C^1(D^\delta)$, V is nonnegative, and there exists some constant $\lambda > 0$ such that $\lambda \leq V(x) \leq \lambda^{-1}$ for $x \in D^{4\delta}$.
- (V2) For a small constant $\delta > 0$, $V(x) \in C(\mathbb{R}^N; \mathbb{R}^+) \cap C^1(D^\delta)$ and there exist some $\alpha > 0$ and $\beta \leq 1$ such that $V(x) \geq \alpha|x|^{-2}(\log|x|)^{-\beta}$ for $x \in \mathbb{R}^N \setminus D^{4\delta}$.
- (D) There are a compact set $M \subset D$ and an open set $U \supset M$ such that for sufficiently small $\delta > 0$, $F^\delta : U \rightarrow \mathbb{R}$ with $\|\nabla F - \nabla V\|_{L^\infty(U)} \leq \delta$, there exists a zero point $y^\delta \in U$ of ∇F^δ satisfying $\lim_{\delta \rightarrow 0} \text{dist}(y^\delta, M) = 0$.

The assumption to (D) was given in [9,22].

It is well known (see Kwong [20]) that

$$\begin{cases} \Delta u - u + u^p = 0, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & u(0) = \max_{\mathbb{R}^N} u, \end{cases} \tag{1.4}$$

has a unique positive solution $U(x)$ which is nondegenerate in the sense that the linearized operator $\Delta - 1 + pU^{p-1}$, a map from $H^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$, has an N -dimensional kernel spanned by $\{\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_N}\}$. Moreover, the solution $U(x)$ satisfies that

$$\begin{cases} U(x) = U(|x|), & x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} U(x)e^{|x|} |x|^{\frac{N-1}{2}} = c > 0, & \lim_{|x| \rightarrow \infty} \frac{U'(x)}{U(x)} = -1. \end{cases} \tag{1.5}$$

For some fixed $\varphi \in C_0^\infty(B_{2\delta}(0))$ such that $\varphi \equiv 1$ on $B_\delta(0)$ and $|\nabla \varphi| \leq 4\delta^{-1}$, we define

$$U_{\varepsilon, y}(x) \equiv s\varphi(x-y)V^{\frac{1}{p-1}}(y)U\left(V^{\frac{1}{2}}(y)\frac{x-y}{\varepsilon}\right).$$

Then, we have the following results.

Theorem 1.1. Under the assumption (V1), we assume that there is an isolated set M satisfying (D). Then, for sufficiently small $\varepsilon > 0$, $N \geq 3$, $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ and some $y_M \in M^\delta$, (1.3) has a solution u_ε such that

$$u_\varepsilon(x) = U_{\varepsilon, y_M}(x) + w_{\varepsilon, y_M}(x),$$

where $\lim_{\varepsilon \rightarrow 0} \text{dist}(y_M, M) = 0$. Moreover, for arbitrary constant $\sigma > 0$, there exists some $c > 0$ such that $\|w_{\varepsilon, y_M}\|_{L^\infty(B_{3\delta}(y_M))} \leq \varepsilon$ and

$$w_{\varepsilon, y_M}(x) \leq \exp^{-\frac{c}{\varepsilon}} |y_M - x|^{-N+2} \left\{ (N-2)\sigma - \left(\log \frac{3|y_M - x|}{\delta} \right)^{-\sigma} \right\}, \quad x \in \mathbb{R}^N \setminus B_{3\delta}(y_M).$$

Theorem 1.2. Under the assumption (V2), we assume that there is an isolated set M satisfying (D). Then, for sufficiently small $\varepsilon > 0$, $N \geq 3$, $p = \frac{N}{N-2}$ and some $y_M \in M^\delta$, (1.3) has a solution u_ε such that

$$u_\varepsilon(x) = U_{\varepsilon, y_M}(x) + w_{\varepsilon, y_M}(x),$$

where $\lim_{\varepsilon \rightarrow 0} \text{dist}(y_M, M) = 0$. Moreover, for any constant $\sigma' \geq \frac{N-2}{2}$, there exists $c > 0$ such that $\|w_{\varepsilon, y_M}\|_{L^\infty(B_{3\delta}(y_M))} \leq \varepsilon$ and

$$w_{\varepsilon, y_M}(x) \leq \exp^{-\frac{c}{\varepsilon}} |y_M - x|^{-N+2} \left(\log \frac{3|y_M - x|}{\delta} \right)^{-\sigma'}, \quad x \in \mathbb{R}^N \setminus B_{3\delta}(y_M).$$

Let H_ε be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon^2 = \langle u, u \rangle_\varepsilon = \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + V(x)u^2 dx.$$

Since H_ε may not be contained in $L^{p+1}(\mathbb{R}^N)$, we will formerly work in the bounded domain $B_\tau(0)$, where $\tau > 2R$ is an arbitrary constant and R is a sufficiently large constant such that $D_{4\delta} \subset B_R(0)$. We define $H_{\varepsilon, \tau}$ the completion of $C_0^\infty(B_\tau(0))$ with respect to the restricted norm

$$\|u\|_{\varepsilon, \tau}^2 = \langle u, u \rangle_{\varepsilon, \tau} = \int_{B_\tau(0)} \varepsilon^2 |\nabla u|^2 + V(x)u^2 dx.$$

In fact, $H_{\varepsilon, \tau}$ is simply $H_0^1(B_\tau(0))$ with a different but equivalent norm, where the equivalence is not uniform in ε . We define a subspace $E_{\varepsilon, y, \tau}$ of $H_{\varepsilon, \tau}$, such that

$$E_{\varepsilon, y, \tau} = \left\{ w \in H_{\varepsilon, \tau} \mid \left\langle w, \frac{\partial U_{\varepsilon, y}}{\partial x_i} \right\rangle_{\varepsilon, \tau} = 0, i = 1, \dots, N \right\}.$$

For $u \in H_{\varepsilon, \tau}$, let

$$I_{\varepsilon, \tau}(u) = \frac{1}{2} \int_{B_\tau(0)} \varepsilon^2 |\nabla u|^2 + V(x)u^2 dx - \frac{1}{p+1} \int_{B_\tau(0)} u^{p+1} dx.$$

We will use Lyapunov–Schmidt reduction method as in [21]. First, for each sufficiently large $\tau > 0$ and $y \in M^\delta$ we will solve that

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + |u|^{p-1}u = 0, & x \in B_\tau(0), \\ u(x) = 0, & x \in \partial B_\tau(0). \end{cases} \tag{1.6}$$

From the contraction argument, we can find a solution $U_{\varepsilon, y} + w_{\varepsilon, y, \tau}$ of an auxiliary equation. Then, from the Pohozaev identity and the assumption (D), we can deduce that there exists $y' \in M^\delta$ such that $I'_{\varepsilon, \tau}(U_{\varepsilon, y'} + w_{\varepsilon, y', \tau}) = 0$. Therefore we find a critical point $u_{\varepsilon, \tau}$ of the energy functional $I_{\varepsilon, \tau}$ and then, from our construction, $u_{\varepsilon, \tau}$ converges to a solution u_ε of (1.3).

Remark 1.3. In the similar way of [21], Theorem 1.1 and Theorem 1.2 can be extended to the existence of multi-bump solutions which allow some bumps concentrating on the zero set of $V(x)$.

2. Proofs of Theorem 1.1 and Theorem 1.2

To apply Lyapunov–Schmidt reduction method, we need the invertibility about the auxiliary equation. Since $U_{\varepsilon, y}$ has a compact support and $V(x)$ is positive in the support of $U_{\varepsilon, y}$, we can obtain the following lemma as in the proofs of Lemma 3.1 in [9], Lemma 2.1 in [12] and Lemma 2.1 in [21].

Lemma 2.1. *There exists a constant $c > 0$, independent of τ and ε , such that for each $\tau > 2R$, $y \in M^\delta$, sufficiently small $\varepsilon > 0$ and $w \in E_{\varepsilon, y, \tau}$,*

$$|I''_{\varepsilon, \tau}(U_{\varepsilon, y})[w, w]| \geq c \|w\|_{\varepsilon, \tau}^2.$$

Proof. Assume by contradiction that there exist sequences $\{\varepsilon_n\}$, $\varepsilon_n \rightarrow 0$, $\{\tau_n\}$, $\tau_n \rightarrow \infty$, $\{y_n\} \subset \mathbb{R}^N$, $y_n \rightarrow y \in M^\delta$ as $n \rightarrow \infty$ and $w_{\varepsilon_n} \in E_{\varepsilon_n, y_n, \tau_n} \setminus \{0\}$ such that

$$\int_{B_{\tau_n}(0)} \varepsilon_n^2 |\nabla w_{\varepsilon_n}|^2 + V(x)w_{\varepsilon_n}^2 dx - p \int_{B_{\tau_n}(0)} U_{\varepsilon_n, y_n}^{p-1} w_{\varepsilon_n}^2 dx = o(1) \|w_{\varepsilon_n}\|_{\varepsilon_n, \tau_n}^2. \tag{2.1}$$

For convenience's sake, we write w_ε , τ , y , $U_{\varepsilon, y}$ for w_{ε_n} , τ_n , y_n , U_{ε_n, y_n} , respectively.

We claim that

$$\int_{B_{2\delta}(y)} U_{\varepsilon,y}^{p-1} w_\varepsilon^2 dx = o(1) \|w_\varepsilon\|_{\varepsilon,\tau}^2.$$

If it is true, then since the support of $U_{\varepsilon,y}$ is contained in $B_{2\delta}(y)$, it follows that

$$\int_{B_{\tau\eta}(0)} \varepsilon^2 |\nabla w_\varepsilon|^2 + V(x) w_\varepsilon^2 - p U_{\varepsilon,y,\tau}^{p-1} w_\varepsilon^2 dx = (1 - o(1)) \|w_\varepsilon\|_{\varepsilon,\tau}^2$$

which contradicts (2.1). Therefore it is enough to prove the claim, then the proof is complete.

We define $v_\varepsilon(x) = w_\varepsilon(\varepsilon x + y)$ and define a norm

$$\|v_\varepsilon\|_\varepsilon^2 = \int_{B_{\tau/\varepsilon}(-y/\varepsilon)} |\nabla v_\varepsilon|^2 + V(\varepsilon x + y) v_\varepsilon^2 dx.$$

Without loss of generality, we can assume $\|v_\varepsilon\|_\varepsilon = 1$. Then, since $\inf_{x \in B_{2\delta}(y)} V(x)$ is positive, we can say that $v_\varepsilon \cdot \psi_{B_{2\delta/\varepsilon}(0)}$ is bounded in $H^1(\mathbb{R}^N)$, where $\psi_A \in C_0^\infty(A^1)$ satisfies that $\psi_A \equiv 1$ on A and $|\nabla \psi_A| \leq 4$ for any $A \subset \mathbb{R}^N$. Therefore, there exists a function $v \in H^1(\mathbb{R}^N)$ such that if we take a subsequence $v_\varepsilon \cdot \psi_{B_{2\delta/\varepsilon}(0)} \rightarrow v$ in $H^1(\mathbb{R}^N)$ and $v_\varepsilon \cdot \psi_{B_{2\delta/\varepsilon}(0)} \rightarrow v$ in $L^2_{loc}(\mathbb{R}^N)$. It follows that

$$\int_{\mathbb{R}^N} \nabla v \cdot \nabla \eta + V(y_M) v \eta - p U_{y_M}^{p-1} v \eta dx = 0,$$

where $U_{y_M} = U(V^{1/2}(y_M)x)$ and $\int_{\mathbb{R}^N} \nabla \frac{\partial U_{y_M}}{\partial x_i} \cdot \nabla \eta + V(y_M) \frac{\partial U_{y_M}}{\partial x_i} \eta dx = 0$. From the nondegeneracy of U_{y_M} , we can write $v = \sum_{i=1}^N a_i \frac{\partial U_{y_M}}{\partial x_i}$ for some $a_i \in \mathbb{R}$. But since $w_\varepsilon \in E_{\varepsilon,y,\tau}$, we obtain that for each $i = 1, \dots, N$,

$$0 = \int_{\mathbb{R}^N} \nabla \frac{\partial U_{y_M}}{\partial x_i} \cdot \nabla v + V(y_M) \frac{\partial U_{y_M}}{\partial x_i} v dx,$$

and this means that $v_\varepsilon \rightarrow 0$ in $L^2(B_{2\delta/\varepsilon}(0))$. Moreover, $\|w_\varepsilon\|_{\varepsilon,\tau}^2 = \varepsilon^N \|v_\varepsilon\|_\varepsilon^2 = \varepsilon^N$. From the exponential decay of U_y , we obtain that

$$\int_{B_\delta(y)} U_{\varepsilon,y}^{p-1} w_\varepsilon^2 dx \leq o(\varepsilon^N) \int_{B_{\delta/\varepsilon}(0)} U_y^{p-1} v_\varepsilon^2 dy = o(\varepsilon^N).$$

Therefore, it follows that $\int_{B_\delta(y)} U_{\varepsilon,y,\tau}^{p-1} w_\varepsilon^2 dx = o(1) \|w_\varepsilon\|_{\varepsilon,\tau}^2$.

Therefore our proof is complete. \square

Now we consider the auxiliary equation. For convenience's sake, we denote $A_\delta(y) \equiv B_{3\delta}(y) \setminus B_{2\delta}(y)$. To study the auxiliary equation we define a set $S_{\varepsilon,y,\tau}$ by

$$S_{\varepsilon,y,\tau} := \left\{ w \in E_{\varepsilon,y,\tau} \mid \|w\|_{\varepsilon,\tau} \leq C_1 \varepsilon^{\frac{N}{2}+1}, \|w\|_{L^\infty(B_{3\delta}(y))} \leq \varepsilon, \right. \\ \left. |w(x)| \leq \varepsilon \exp^{\frac{c_1}{\varepsilon}(|x-y|-2\delta)(|x-y|-3\delta)}, x \in A_\delta(y) \text{ and} \right. \\ \left. |w(x)| \leq \exp^{-\frac{c}{\varepsilon}} |y-x|^{-N+2} \left\{ (N-2)\sigma - \left(\log \frac{3|y-x|}{\delta} \right)^{-\sigma} \right\}, x \in B_\tau(0) \setminus B_{3\delta}(y) \right\},$$

where C_1, c_1 and c are constants to be determined.

Proposition 2.2. For each $\tau > 2R$, sufficiently small $\varepsilon > 0$, and $y \in M^\delta$, there exist some constant $C_{i,\varepsilon,y} = O(\varepsilon^2)$ and a unique $w_{\varepsilon,y,\tau} \in S_{\varepsilon,y,\tau}$ satisfying

$$I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau}) = \sum_{i=1}^N C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i}.$$

Proof. Since $U_{\varepsilon,y,\tau}$ has an exponential decay rate and is bounded in L^∞ , for any $w \in S_{\varepsilon,y,\tau}$, $I'_{\varepsilon,\tau}(U_{\varepsilon,y,\tau} + w)[\cdot]$ and $I''_{\varepsilon,\tau}(U_{\varepsilon,y,\tau})[w, \cdot]$ are bounded linear functional on $H_{\varepsilon,\tau}$. Therefore, by the Riesz representation theorem, there are maps $PI'_{\varepsilon,\tau}(U_{\varepsilon,y} + \cdot)$ and $PI''_{\varepsilon,\tau}(U_{\varepsilon,y})[\cdot]$ from $E_{\varepsilon,y,\tau}$ to itself. We define a map $G_{\varepsilon,\tau}$ such that for any $w \in E_{\varepsilon,y,\tau}$,

$$G_{\varepsilon,\tau}(w) \equiv -[PI''_{\varepsilon,\tau}(U_{\varepsilon,y})]^{-1}[PI'_{\varepsilon,\tau}(U_{\varepsilon,y} + w) - PI''_{\varepsilon,\tau}(U_{\varepsilon,y})[w]],$$

and from Lemma 2.1 this is well defined. If we could show that the set $S_{\varepsilon,y,\tau}$ has a fixed point $w_{\varepsilon,y,\tau}$ for the map $G_{\varepsilon,\tau}$, then $U_{\varepsilon,y} + w_{\varepsilon,y,\tau}$ is a solution of the auxiliary equation, i.e., $PI'_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau}) = 0$. Using the contraction argument, we can find the fixed point.

Step 1. We claim that for any $w \in S_{\varepsilon,y,\tau}$,

$$\|G_{\varepsilon,\tau}(w)\|_{\varepsilon,\tau} \leq C_1 \varepsilon^{\frac{N}{2}+1}$$

and for some $0 < \theta < 1$ and any $w, \tilde{w} \in S_{\varepsilon,y,\tau}$,

$$\|G_{\varepsilon,\tau}(w) - G_{\varepsilon,\tau}(\tilde{w})\|_{\varepsilon,\tau} \leq \theta \|w - \tilde{w}\|_{\varepsilon,\tau}.$$

From Lemma 2.1, it follows that for some $c_0 > 0$ and $\eta \in E_{\varepsilon,y,\tau}$,

$$\begin{aligned} \|G_{\varepsilon,\tau}(w)\|_{\varepsilon,\tau} &\leq c_0 \|PI''_{\varepsilon,\tau}(U_{\varepsilon,y})[G_{\varepsilon,\tau}(w)]\|_{\varepsilon,\tau} \\ &= \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} c_0 |I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w)[\eta] - I''_{\varepsilon,\tau}(U_{\varepsilon,y})[w, \eta]| \\ &\leq \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} c_0 \left| \int_{B_\tau(0)} \{-\varepsilon^2 \Delta(U_{\varepsilon,y} + w) + V(x)(U_{\varepsilon,y} + w) \right. \\ &\quad \left. - (U_{\varepsilon,y} + w)^p + \varepsilon^2 \Delta w - V(x)w + pU_{\varepsilon,y}^{p-1} w\} \eta \, dx \right| \\ &\leq \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} c_0 \left| \int_{B_\tau(0)} \{-\varepsilon^2 \Delta U_{\varepsilon,y} + V(x)U_{\varepsilon,y} - U_{\varepsilon,y}^p\} \eta \, dx \right. \\ &\quad \left. - \int_{B_\tau(0)} \{(U_{\varepsilon,y} + w)^p - U_{\varepsilon,y}^p - pU_{\varepsilon,y}^{p-1} w\} \eta \, dx \right| \\ &\equiv E_1 + E_2. \end{aligned}$$

If we take $r = V^{\frac{1}{2}}(y) \frac{|x-y|}{\varepsilon}$ and $c_2 = \sup_{B_{2\delta}(y)} |\nabla V(x)|$, then it follows that

$$\begin{aligned} \int_{B_{2\delta}(y)} (V(x) - V(y))^2 U_{\varepsilon,y}^2 \, dx &\leq \int_{B_{2\delta}(y)} c_2^2 |x-y|^2 U_{\varepsilon,y}^2 \, dx \\ &\leq c_2^2 \omega_N V^{\frac{4}{p-1} - \frac{N}{2}}(y) \varepsilon^{N+2} \int_0^{2\delta\varepsilon^{-1}} r^2 U^2(r) r^{N-1} \, dr \\ &\leq c_2^2 c_3^2 \omega_N V^{\frac{4}{p-1} - \frac{N}{2}}(y) \varepsilon^{N+2} \int_0^{2\delta\varepsilon^{-1}} r^2 \exp^{-2r} \, dr \\ &\leq c_2^2 c_3^2 \omega_N V^{\frac{4}{p-1} - \frac{N}{2}}(y) \varepsilon^{N+2}, \end{aligned}$$

where a constant c_3 satisfies $U(r) \leq c_3 \exp^{-r} r^{-\frac{N-1}{2}}$ for any $r \geq 0$. Since $U_{\varepsilon,y,\tau}$ has an exponentially decay rate and $\inf_{x \in B_{2\delta}(y)} V(x) = \lambda > 0$, as in the proof of Proposition 2.2 in [21], we obtain that

$$\begin{aligned} E_1 &\leq c_0 \int_{B_\tau(0)} |(\varepsilon^2 \Delta U_{\varepsilon,y} - V(x)U_{\varepsilon,y} + U_{\varepsilon,y}^p)\eta| \, dx \\ &\leq c_0 \lambda^{-\frac{1}{2}} \left(\int_{B_{2\delta}(y)} (V(x) - V(y))^2 U_{\varepsilon,y}^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_{2\delta}(y)} V(x) \eta^2 \, dx \right)^{\frac{1}{2}} + o(\varepsilon^{\frac{N}{2}+1}) \|\eta\|_{\varepsilon,\tau} \leq \frac{3}{4} C_1 \varepsilon^{\frac{N}{2}+1} \|\eta\|_{\varepsilon,\tau}, \end{aligned} \tag{2.2}$$

where $C_1 = 2\lambda_1 c_0 c_2 c_3 c_4 w_N^{\frac{1}{2}}$ and $c_4 = \sup_{x \in B_{2\delta}(y)} V^{\frac{2}{p-1} - \frac{N}{4}}(y)$.

Let $\bar{p} = \min\{p, 2\}$. Then we get that for some $C > 0$, independent of ε ,

$$E_2 \leq \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} c_0 \left| \int_{B_\tau(0)} \{(U_{\varepsilon,y} + w)^p - U_{\varepsilon,y}^p - pU_{\varepsilon,y}^{p-1}w\} \eta dx \right| \leq C\varepsilon^{(\bar{p}-1)} \int_{B_{3\delta}(y)} |w||\eta| dx + \int_{B_\tau(0) \setminus B_{3\delta}(y)} |w|^p |\eta| dx.$$

Since $\inf_{x \in B_{3\delta}(y)} V(x) > 0$,

$$C\varepsilon^{(\bar{p}-1)} \int_{B_{3\delta}(y)} |w||\eta| dx < C'\varepsilon^{\frac{N}{2} + \bar{p}} \|\eta\|_{\varepsilon,\tau}.$$

Using the Poincaré inequality and the decay rate of $w(x)$ in $B_\tau(0) \setminus B_{3\delta}$, we obtain that

$$\begin{aligned} \int_{B_\tau(0) \setminus B_{3\delta}} |w|^p |\eta| dx &\leq \|w\|_{L^{\frac{2N}{N-2}}(B_\tau(0) \setminus B_{3\delta})} \|\eta\|_{L^{\frac{2N}{N-2}}(B_\tau(0) \setminus B_{3\delta})} \left(\int_{B_\tau(0) \setminus B_{3\delta}} |w|^{(p-1)\frac{N}{2}} dx \right)^{\frac{2}{N}} \\ &\leq C\varepsilon^{-2} \exp^{-\frac{c(p-1)}{\varepsilon}} \|w\|_{\varepsilon,\tau} \|\eta\|_{\varepsilon,\tau} \left(\int_{B_\tau(0) \setminus B_{3\delta}} |y-x|^{-\frac{(p-1)(N-2)N}{2}} dx \right)^{\frac{2}{N}}. \end{aligned}$$

For sufficiently small ε , this implies that

$$E_2 \leq \frac{1}{4} \|w\|_{\varepsilon,\tau} \|\eta\|_{\varepsilon,\tau} \leq \frac{1}{4} C_1 \varepsilon^{\frac{N}{2} + 1} \|\eta\|_{\varepsilon,\tau}. \tag{2.3}$$

Therefore we obtain from (2.2) and (2.3) that

$$\|G_{\varepsilon,\tau}(w)\|_{\varepsilon,\tau} \leq C_1 \varepsilon^{\frac{N}{2} + 1}.$$

Similarly as in the previous calculation, we obtain that for any $w, \tilde{w} \in S_{\varepsilon,y,\tau}$,

$$\begin{aligned} \|G_{\varepsilon,\tau}(w) - G_{\varepsilon,\tau}(\tilde{w})\|_{\varepsilon,\tau} &\leq \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} C |I''_{\varepsilon,\tau}(U_{\varepsilon,y})[G_{\varepsilon,\tau}(w) - G_{\varepsilon,\tau}(\tilde{w}), \eta]| \\ &= \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} C |I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w)[\eta] - I'_{\varepsilon,\tau}(U_{\varepsilon,y} + \tilde{w})[\eta] \\ &\quad - I'_{\varepsilon,\tau}(U_{\varepsilon,y})[w, \eta] + I'_{\varepsilon,\tau}(U_{\varepsilon,y})[\tilde{w}, \eta]| \\ &\leq \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} C \int_{B_\tau(0)} | \{- (U_{\varepsilon,y} + w)^p + (U_{\varepsilon,y} + \tilde{w})^p + pU_{\varepsilon,y}^{p-1}w - pU_{\varepsilon,y}^{p-1}\tilde{w} \} \eta | dx \\ &\leq \sup_{\|\eta\|_{\varepsilon,\tau} \leq 1} C \int_{B_\tau(0)} |w - \tilde{w}| \{ |w|^{p-1} + |\tilde{w}|^{p-1} + U_{\varepsilon,y}^{p-2}(|w| + |\tilde{w}|) \} |\eta| dx \\ &= o(1) \|w - \tilde{w}\|_{\varepsilon,\tau}. \end{aligned}$$

Therefore, we deduce that for some $0 < \theta < 1$ and any $w, \tilde{w} \in S_{\varepsilon,y,\tau}$,

$$\|G_{\varepsilon,\tau}(w) - G_{\varepsilon,\tau}(\tilde{w})\|_{\varepsilon,\tau} \leq \theta \|w - \tilde{w}\|_{\varepsilon,\tau}.$$

This completes Step 1.

Step 2. We claim that for any $w \in S_{\varepsilon,y,\tau}$,

$$G_{\varepsilon,\tau}(w) \in S_{\varepsilon,y,\tau}.$$

By the definition of $G_{\varepsilon,\tau}$, we can immediately check that $G_{\varepsilon,\tau}(w) \in E_{\varepsilon,y,\tau}$.

For $w \in S_{\varepsilon,y,\tau}$, we define

$$\tilde{w} \equiv G_{\varepsilon,\tau}(w).$$

Then by Lemma 2.1,

$$PI''_{\varepsilon,\tau}(U_{\varepsilon,y})[\tilde{w}] = -PI'_{\varepsilon,\tau}(U_{\varepsilon,y} + w) + PI''_{\varepsilon,\tau}(U_{\varepsilon,y})[w].$$

This implies that for some $C_{i,\varepsilon,y} \in \mathbb{R}$ and each $j = 1, \dots, N$,

$$I''_{\varepsilon,\tau}(U_{\varepsilon,y}) \left[\tilde{w}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right] = -I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w) \left[\frac{\partial U_{\varepsilon,y}}{\partial x_j} \right] + I''_{\varepsilon,\tau}(U_{\varepsilon,y}) \left[w, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right] + \sum_{i=1}^N C_{i,\varepsilon,y} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_{\varepsilon,\tau}.$$

We claim that $C_{i,\varepsilon,y} = O(\varepsilon^2)$ for $i = 1, \dots, N$.

For each $j = 1, \dots, N$, we see that

$$\begin{aligned} \sum_{i=1}^N C_{i,\varepsilon,y} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\rangle_{\varepsilon,\tau} &= \int_{B_\tau(0)} (-\varepsilon^2 \Delta \tilde{w} + V(x) \tilde{w} - p U_{\varepsilon,y}^{p-1} \tilde{w}) \frac{\partial U_{\varepsilon,y}}{\partial x_j} dx \\ &\quad + \int_{B_\tau(0)} (-\varepsilon^2 \Delta U_{\varepsilon,y} + V(x) U_{\varepsilon,y} - U_{\varepsilon,y}^p) \frac{\partial U_{\varepsilon,y}}{\partial x_j} dx \\ &\quad - \int_{B_\tau(0)} \{(U_{\varepsilon,y} + w)^p - p U_{\varepsilon,y}^{p-1} w - U_{\varepsilon,y}^p\} \frac{\partial U_{\varepsilon,y}}{\partial x_j} dx. \end{aligned}$$

By the same computation as for equation E_1 in Step 1, we see that from $\|\frac{\partial U_{\varepsilon,y}}{\partial x_j}\|_{\varepsilon,\tau} = O(\varepsilon^{\frac{N}{2}-1})$

$$\begin{aligned} &\int_{B_\tau(0)} \left| (\varepsilon^2 \Delta U_{\varepsilon,y} - V(x) U_{\varepsilon,y} + U_{\varepsilon,y}^p) \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right| dx + \int_{B_\tau(0)} \left| \{(U_{\varepsilon,y} + w)^p - p U_{\varepsilon,y}^{p-1} w - U_{\varepsilon,y}^p\} \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right| dx \\ &\leq C_1 \varepsilon^{\frac{N}{2}+1} \left\| \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\|_{\varepsilon,\tau} \leq O(\varepsilon^N). \end{aligned}$$

Moreover, from the exponential decay property of $U_{\varepsilon,y}$, we see that for some $C > 0$

$$\begin{aligned} &\int_{B_\tau(0)} (-\varepsilon^2 \Delta \tilde{w} + V(x) \tilde{w} - p U_{\varepsilon,y}^{p-1} \tilde{w}) \frac{\partial U_{\varepsilon,y}}{\partial x_j} dx \\ &= \int_{B_{2\delta}(0)} \left(-\varepsilon^2 \Delta \frac{\partial U_{\varepsilon,y}}{\partial x_j} + V(x) \frac{\partial U_{\varepsilon,y}}{\partial x_j} - p U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right) \tilde{w} dx + o(\varepsilon^N) \\ &= \int_{B_{2\delta}(0)} (V(x) - V(y)) \frac{\partial U_{\varepsilon,y}}{\partial x_j} \tilde{w} + p (U_{\varepsilon,y}^{p-1} - U_{\varepsilon,y}^{p-1}) \frac{\partial U_{\varepsilon,y}}{\partial x_j} \tilde{w} dx + o(\varepsilon^N) \\ &\leq C \|\tilde{w}\|_{\varepsilon,\tau} \left\| \frac{\partial U_{\varepsilon,y}}{\partial x_j} \right\|_{\varepsilon,\tau} \leq O(\varepsilon^N). \end{aligned}$$

Therefore we obtain that $C_{i,\varepsilon,y} = O(\varepsilon^2)$.

For brevity, letting

$$f_{\varepsilon,y}(x) \equiv \sum_{i=1}^N C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i} - I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w) + I''_{\varepsilon,\tau}(U_{\varepsilon,y})[w],$$

we see that

$$-\varepsilon^2 \Delta \tilde{w} + V(x) \tilde{w} - p U_{\varepsilon,y}^{p-1} \tilde{w} = f_{\varepsilon,y}(x).$$

From the exponential decay property of $U_{\varepsilon,y}$, we obtain that

$$|f_{\varepsilon,y}(x)| \leq \varepsilon^p \exp^{\frac{c_1 p}{\varepsilon} (|x-y|-2\delta)(|x-y|-3\delta)}, \quad x \in A_\delta(y)$$

and

$$|f_{\varepsilon,y}(x)| \leq \exp^{-\frac{c_2 p}{\varepsilon}} |y-x|^{-p(N-2)} \left\{ (N-2)\sigma - \left(\log \frac{3|y-x|}{\delta} \right)^{-\sigma} \right\}^p, \quad x \in B_\tau(0) \setminus B_{3\delta}(y).$$

Using these estimates we will obtain the estimates of \tilde{w} adjusted to the set $S_{\varepsilon,y,\tau}$.

First, we will show that $\|\tilde{w}\|_{L^\infty(B_{3\delta}(y))} \leq \varepsilon$. We see that

$$\|\tilde{w}\|_{L^2(B_{4\delta}(y))} \leq \lambda^{-1} \|\tilde{w}\|_{\varepsilon,\tau} \leq \lambda^{-1} C_1 \varepsilon^{\frac{N}{2}+1},$$

and for some C and $C' > 0$, independent of ε ,

$$\|f_{\varepsilon,y}\|_{L^N(B_{4\delta}(y))} \leq \sum_{i=1}^N C \varepsilon^2 \left\| \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\|_{L^N(B_{4\delta}(y))} + \|(V(x) - V(y)) U_{\varepsilon,y}\|_{L^N(B_{4\delta}(y))} + C \|w^{\bar{p}}\|_{L^N(B_{4\delta}(y))} \leq C' \varepsilon^{\bar{p}}.$$

Therefore, from the Moser iteration theorem (for example, see Theorem 9.20 [18]), it follows that for any $z \in B_{2\delta}(y)$

$$\begin{aligned} \|\tilde{w}\|_{L^\infty(B_\delta(z))} &\leq C_2(\delta^{-\frac{N}{2}}\|\tilde{w}\|_{L^2(B_{2\delta}(z))} + \delta\|f_{\varepsilon,y}\|_{L^N(B_{2\delta}(z))}) \\ &\leq C_2(\delta^{-\frac{N}{2}}\|\tilde{w}\|_{L^2(B_{4\delta}(y))} + \delta\|f_{\varepsilon,y}\|_{L^N(B_{4\delta}(y))}), \end{aligned}$$

where the constant C_2 depends only N and δ . This implies that $\|\tilde{w}\|_{L^\infty(B_{3\delta}(y))} \leq \varepsilon$.

Second, we will show that $\|\tilde{w}\|_{L^\infty(A_\delta)} \leq \varepsilon \exp^{\frac{c_1}{\varepsilon}(|x-y|-2\delta)(|x-y|-3\delta)}$ in A_δ , where $c_1 = \frac{\lambda^{1/2}}{2\delta}$. Let

$$\Psi(x) \equiv \varepsilon \exp^{\frac{c_1}{\varepsilon}(|x-y|-2\delta)(|x-y|-3\delta)},$$

then

$$-\varepsilon^2 \Delta \Psi(x) + V(x)\Psi(x) = \left\{ -c_1^2(2|x-y|-5\delta)^2 - 2c_1\varepsilon - \frac{c_1\varepsilon|x-y|(N-1)}{|x-y|} + V(x) \right\} \Psi(x) \geq \frac{\lambda}{2} \Psi(x)$$

and $\Psi(x) = \varepsilon$, $x \in \partial A_\delta$. Therefore, $-\varepsilon^2 \Delta(\Psi(x) - \tilde{w}) + V(x)(\Psi(x) - \tilde{w}) \geq \frac{\lambda}{2} \Psi(x) - w^p \geq \frac{\lambda}{4} \Psi(x) > 0$. By the maximum principle, we obtain that $|\tilde{w}(x)| \leq \Psi(x)$ for any $x \in A_\delta$, and $|\tilde{w}(x)| \leq \exp^{-\frac{\lambda^{1/2}\delta}{8\varepsilon}}$, $x \in B_{\frac{5}{2}\delta}(y)$.

Finally, to obtain L^∞ estimate for \tilde{w} in $B_\tau(0) \setminus B_{3\delta}(y)$, let $\Psi_1(x) = \exp^{-\frac{\varepsilon}{\delta}}|y-x|^{-N+2}\{(N-2)\sigma - (\log \frac{3|y-x|}{\delta})^{-\sigma}\}$, where $c = \frac{\lambda^{1/2}\delta}{4}$ (see [23]). Since $\text{supp}\{U_{\varepsilon,y}\} \in B_{2\delta}(y)$, we obtain that, in $B_\tau(0) \setminus B_{\frac{5}{2}\delta}(y)$,

$$\begin{aligned} &-\varepsilon^2 \Delta(\Psi_1(x) - \tilde{w}) + V(x)(\Psi_1(x) - \tilde{w}) \\ &= \varepsilon^2 \exp^{-\frac{\varepsilon}{\delta}} \sigma |y-x|^{-N} \left(\log \frac{3|y-x|}{\delta} \right)^{-\sigma-1} \left\{ N-2 + (\sigma+1) \left(\log \frac{3|y-x|}{\delta} \right)^{-1} \right\} + V(x)\Psi_1 - w^p \geq 0 \end{aligned} \tag{2.4}$$

and

$$|\tilde{w}(x)| \leq \Psi_1(x), \quad x \in \partial B_{\frac{5}{2}\delta}(y).$$

Therefore, by the maximum principle, we obtain that $|\tilde{w}(x)| \leq \Psi_1(x)$ for any $x \in B_\tau(0) \setminus B_{\frac{5}{2}\delta}(y)$, and $|\tilde{w}(x)| \leq \exp^{-\frac{\varepsilon}{\delta}}|y-x|^{-N+2}\{(N-2)\sigma - (\log \frac{3|y-x|}{\delta})^{-\sigma}\}$, in $B_\tau(0) \setminus B_{3\delta}(y)$.

Therefore, it follows that

$$\tilde{w} = G_{\varepsilon,\tau}(w) \in S_{\varepsilon,y,\tau}.$$

From the fixed point theorem we see that there exist $w_{\varepsilon,y} \in S_{\varepsilon,y,\tau}$ such that $I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau}) = \sum_{i=1}^N C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i}$ for some $C_{i,\varepsilon,y} \in \mathbb{R}$. \square

For $w \in S_{\varepsilon,y,\tau}$, since $w \in L^q(B_\tau(0))$ for $q > \frac{N+2}{N-2}$ and $\tilde{w} \in L^\infty$, from $W_{\text{loc}}^{2,q}$ estimation, we obtain that $U_{\varepsilon,y} + w_{\varepsilon,y,\tau} \in W_{\text{loc}}^{2,q}(B_\tau(0))$ for $q > \frac{N+2}{N-2}$. To study the bifurcation equation, we estimate $\frac{\partial w_{\varepsilon,y,\tau}}{\partial y_j}$ for $j = 1, \dots, N$.

Lemma 2.3. For each $\tau > 0$, sufficiently small $\varepsilon > 0$ and $y \in M^\delta$, the following estimate holds

$$\left\| \frac{\partial w_{\varepsilon,y,\tau}}{\partial y_j} \right\|_{\varepsilon,\tau} = o(\varepsilon^{\frac{N}{2}-1}). \tag{2.5}$$

Proof. From Proposition 2.2 we see that for any $v \in E_{\varepsilon,y,\tau}$,

$$\int_{B_\tau(0)} \varepsilon^2 \nabla(U_{\varepsilon,y} + w_{\varepsilon,y,\tau}) \nabla v + V(x)(U_{\varepsilon,y} + w_{\varepsilon,y,\tau})v - (U_{\varepsilon,y} + w_{\varepsilon,y,\tau})^p v \, dx = \left\langle \sum_{i=1}^N C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i}, v \right\rangle_{\varepsilon,\tau} = 0.$$

This implies that

$$\begin{aligned} \left\langle \frac{\partial w_{\varepsilon,y,\tau}}{\partial y_j}, v \right\rangle_{\varepsilon,\tau} &= \frac{\partial V(y)}{\partial y_j} \int_{B_\tau(0)} U_{\varepsilon,y} v \, dx - p \int_{B_\tau(0)} (U_{\varepsilon,y}^{p-1} - (U_{\varepsilon,y} + w_{\varepsilon,y,\tau})^{p-1}) \frac{\partial U_{\varepsilon,y}}{\partial y_j} v \, dx \\ &\quad - \int_{B_\tau(0)} (V(x) - V(y)) \frac{\partial U_{\varepsilon,y}}{\partial y_i} v \, dx + p \int_{B_\tau(0)} (U_{\varepsilon,y} + w_{\varepsilon,y,\tau})^{p-1} \frac{\partial w_{\varepsilon,y,\tau}}{\partial y_j} v \, dx + o(\varepsilon^N) \|v\|_{\varepsilon,\tau}. \end{aligned}$$

We can write $\frac{\partial w_{\varepsilon,y,\tau}}{\partial y_j} = w_1 + w_2$ so that $\langle w_1, w_2 \rangle_{\varepsilon,\tau} = 0$, $w_1 \in (E_{\varepsilon,y,\tau})^\perp$ and $w_2 \in E_{\varepsilon,y,\tau}$. Since $w_{\varepsilon,y,\tau} \in E_{\varepsilon,y,\tau}$, we obtain that

$$\left| \left\langle w_1, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_{\varepsilon,\tau} \right| = \left| \left\langle \frac{\partial w_{\varepsilon,y,\tau}}{\partial y_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_{\varepsilon,\tau} \right| = \left| \left\langle w_{\varepsilon,y,\tau}, \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \right\rangle_{\varepsilon,\tau} \right| \leq O(\varepsilon^{N-1})$$

for each $i = 1, \dots, N$, and so $\|w_1\|_{\varepsilon,\tau} \leq O(\varepsilon^{\frac{N}{2}})$. For w_2 , we deduce from Lemma 2.1 and by a similar calculation as in Proposition 2.2 that for some $c > 0$,

$$\begin{aligned} \|w_2\|_{\varepsilon,\tau} \leq & \sup_{\|v\|_{\varepsilon,\tau} \leq 1} c \left| \frac{\partial V(y)}{\partial y_i} \int_{B_\tau(0)} U_{\varepsilon,y} v \, dx - p \int_{B_\tau(0)} \{U_{\varepsilon,y}^{p-1} - (U_{\varepsilon,y} + w_{\varepsilon,y,\tau})^{p-1}\} \frac{\partial U_{\varepsilon,y}}{\partial y_j} v \, dx \right. \\ & \left. - \int_{B_\tau(0)} (V(x) - V(y)) \frac{\partial U_{\varepsilon,y}}{\partial y_j} v \, dx \right| + o(1) \|w_2\|_{\varepsilon,\tau} + O(\varepsilon^{\frac{N}{2}}). \end{aligned}$$

And this implies that $\|w_2\|_{\varepsilon,\tau} = o(\varepsilon^{\frac{N}{2}-1})$. Therefore we get $\|\frac{\partial w_{\varepsilon,y,\tau}}{\partial y_i}\|_{\varepsilon,\tau} = o(\varepsilon^{\frac{N}{2}-1})$.

This completes the proof. \square

Until now, we proved the existence of solution of the auxiliary equation for each $\tau > 0$, sufficiently small $\varepsilon > 0$, and $y \in M^\delta$. From the Pohozaev identity and the condition (D), we need to find a solution y' of the N -equation, $C_{i,\varepsilon,y} = 0$, for each $\tau > 0$ and sufficiently small $\varepsilon > 0$, i.e., $U_{\varepsilon,y'} + w_{\varepsilon,y',\tau}$ is a solution of (1.6).

Proposition 2.4. For each $\tau > 0$ and sufficiently small $\varepsilon > 0$, there exists $y' \in M^\delta$ such that

$$I'_{\varepsilon,\tau}(U_{\varepsilon,y'} + w_{\varepsilon,y',\tau}) = 0.$$

Proof. From Lemma 2.3 and as in the proof of Proposition 2.2, we see that

$$\frac{\partial}{\partial y_i} \left\{ \int_{B_\tau(0)} (U_{\varepsilon,y} + w_{\varepsilon,y,\tau})^{p+1} - U_{\varepsilon,y}^{p+1} - (p+1)U_{\varepsilon,y}^p(U_{\varepsilon,y} + w_{\varepsilon,y,\tau} - U_{\varepsilon,y}) \, dx \right\} = o(\varepsilon^N).$$

Moreover, using Pohozaev's identity, we obtain that

$$\begin{aligned} & \frac{\partial}{\partial y_j} \{I_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau})\} \\ &= \frac{\partial}{\partial y_j} \left\{ \int_{B_\tau(0)} \frac{\varepsilon^2}{2} |\nabla U_{\varepsilon,y}|^2 + \frac{1}{2} V(y) U_{\varepsilon,y}^2 - \frac{1}{p+1} U_{\varepsilon,y}^{p+1} \, dx + \frac{1}{2} \int_{B_\tau(0)} (V(x) - V(y)) U_{\varepsilon,y}^2 \, dx + \frac{1}{2} \|w_{\varepsilon,y,\tau}\|_{\varepsilon,\tau}^2 \right. \\ & \quad \left. + I'_{\varepsilon,\tau}(U_{\varepsilon,y})[w_{\varepsilon,y,\tau}] - \frac{1}{p+1} \int_{B_\tau(0)} (U_{\varepsilon,y} + w_{\varepsilon,y,\tau})^{p+1} - U_{\varepsilon,y}^{p+1} - (p+1)U_{\varepsilon,y}^p w_{\varepsilon,y,\tau} \, dx \right\} \\ &= C_3 \varepsilon^N \frac{\partial(V^\theta(y))}{\partial y_j} + o(\varepsilon^N), \end{aligned}$$

where $\theta = \frac{p+1}{p-1} - \frac{N}{2}$, $C_3 = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} U^{p+1}$ and U is the solution of (1.4). This implies that

$$\begin{aligned} \frac{\partial I_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau})}{\partial y_j} &= I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau}) \left[\frac{\partial(U_{\varepsilon,y} + w_{\varepsilon,y,\tau})}{\partial y_j} \right] \\ &= \left\langle \sum_{i=1}^N C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i}, \frac{\partial(U_{\varepsilon,y} + w_{\varepsilon,y,\tau})}{\partial y_j} \right\rangle_{\varepsilon,\tau} = C_3 \varepsilon^N \frac{\partial(V^\theta(y))}{\partial y_j} + o(\varepsilon^N). \end{aligned}$$

Since the assumption (D), there exists some y' in M^δ such that

$$\left\langle \sum_{i=1}^N C_{i,\varepsilon,y'} \frac{\partial U_{\varepsilon,y'}}{\partial x_i}, \frac{\partial U_{\varepsilon,y'}}{\partial y_j} + \frac{\partial w_{\varepsilon,y',\tau}}{\partial y_j} \right\rangle_{\varepsilon,\tau} = 0.$$

Therefore, from (2.5), we deduce that $C_{i,\varepsilon,y'} \equiv 0$ for $i = 1, \dots, N$, which completes the proof. \square

Proof of Theorem 1.1. The existence of a solution $u_{\varepsilon, \tau}$ of (1.6) follows from Propositions 2.2 and 2.4. For each sufficiently large $\tau > 0$, since $u_{\varepsilon, \tau} \in W_{\text{loc}}^{2,q}(B_{\tau}(0))$ for $q > \frac{N+2}{N-2}$, we can find an extension $\bar{u}_{\varepsilon, \tau} \in H_{\varepsilon}(\mathbb{R}^N) \cap W^{2,q}(\mathbb{R}^N)$ from the extension theorem (for example, see Theorem 4.26 [1]), and $\{\bar{u}_{\varepsilon, \tau}\}$ is a uniformly bounded set in H_{ε} . Therefore, we can take a subsequence such that $\bar{u}_{\varepsilon, \tau} \rightharpoonup u_{\varepsilon}$ in H_{ε} and $\bar{u}_{\varepsilon, \tau} \rightarrow u_{\varepsilon}$ in C^1 . Moreover, u_{ε} is a weak solution of (1.3), and from the property of $S_{\varepsilon, y, \tau}$, it has a form of

$$u_{\varepsilon}(x) = U_{\varepsilon, y_M}(x) + w_{\varepsilon, y_M}(x),$$

where $\lim_{\varepsilon \rightarrow 0} \text{dist}(y_M, M) = 0$. Moreover, $\|w_{\varepsilon, y_M}\|_{L^{\infty}(B_{3\delta}(y_M))} \leq \varepsilon$ and

$$w_{\varepsilon, y_M}(x) \leq \exp^{-\frac{c}{\varepsilon}} |y_M - x|^{-N+2} \left\{ (N-2)\sigma - \left(\log \frac{3|y_M - x|}{\delta} \right)^{-\sigma} \right\}, \quad x \in \mathbb{R}^N \setminus B_{3\delta}(y_M).$$

This completes the proof. \square

Since the assumptions (V1) and (V2) are different only in the $\mathbb{R}^N \setminus D^{4\delta}$, the proof of Theorem 1.2 is a little different from the proof of Theorem 1.1.

Proof of Theorem 1.2. The invertibility of $I''_{\varepsilon, \tau}(U_{\varepsilon, y})$ in Lemma 2.1 is also obtained under the assumption (V2). However, we cannot obtain the inequality (2.4) in $S_{\varepsilon, y, \tau}$ when $p = \frac{N}{N-2}$ and $\sigma' \geq \frac{N-2}{2}$. Therefore, we need to define another closed subset to apply contraction argument.

We define a set $T_{\varepsilon, y, \tau}$ by

$$T_{\varepsilon, y, \tau} := \left\{ w \in E_{\varepsilon, y, \tau} \mid \|w\|_{\varepsilon} \leq C_1 \varepsilon^{\frac{N}{2}+1}, \|w\|_{L^{\infty}(B_{3\delta}(y))} \leq \varepsilon, \right. \\ \left. |w(x)| \leq \varepsilon \exp^{\frac{c_1}{\varepsilon}(|x-y|-2\delta)(|x-y|-3\delta)}, x \in A_{\delta}(y) \text{ and} \right. \\ \left. |w(x)| \leq \exp^{-\frac{c}{\varepsilon}} |y-x|^{-N+2} \left(\log \frac{3|y-x|}{\delta} \right)^{-\sigma'}, x \in B_{\tau}(0) \setminus B_{3\delta}(y) \right\}.$$

Then, we can solve the auxiliary equation in the similar way of Proposition 2.2. Since $S_{\varepsilon, y, \tau}$ and $T_{\varepsilon, y, \tau}$ are only different in $B_{\tau}(0) \setminus B_{3\delta}$, we can obtain same calculations of Proposition 2.2 except the following two calculations.

Using the Poincaré inequality and the decay rate of $w(x)$ in $B_{\tau}(0) \setminus B_{3\delta}(y)$, it follows that

$$\int_{B_{\tau}(0) \setminus B_{3\delta}(y)} |w|^p |\eta| dx \leq \|w\|_{L^{\frac{2N}{N-2}}(B_{\tau}(0) \setminus B_{3\delta}(y))} \| \eta \|_{L^{\frac{2N}{N-2}}(B_{\tau}(0) \setminus B_{3\delta}(y))} \left(\int_{B_{\tau}(0) \setminus B_{3\delta}(y)} |w|^{(p-1)\frac{N}{2}} dx \right)^{\frac{2}{N}} \\ \leq C \varepsilon^{-2} \exp^{-\frac{c(p-1)}{\varepsilon}} \|w\|_{\varepsilon, \tau} \| \eta \|_{\varepsilon, \tau} \left(\int_{B_{\tau}(0) \setminus B_{3\delta}(y)} |y-x|^{-N} \left(\log \frac{3|y-x|}{\delta} \right)^{-N\sigma'} dx \right)^{\frac{2}{N}}.$$

Then, we can show that for any $w \in S_{\varepsilon, y, \tau}$,

$$\|G_{\varepsilon, \tau}(w)\|_{\varepsilon, \tau} \leq C_1 \varepsilon^{\frac{N}{2}+1}$$

and for some $0 < \theta < 1$ and any $w, \tilde{w} \in S_{\varepsilon, y, \tau}$,

$$\|G_{\varepsilon, \tau}(w) - G_{\varepsilon, \tau}(\tilde{w})\|_{\varepsilon, \tau} \leq \theta \|w - \tilde{w}\|_{\varepsilon, \tau}.$$

Moreover, to obtain L^{∞} estimate for \tilde{w} in $B_{\tau}(0) \setminus B_{3\delta}(y)$, let $\Psi_2(x) = \exp^{-\frac{c}{\varepsilon}} |y-x|^{-N+2} \left(\log \frac{3|y-x|}{\delta} \right)^{-\sigma'}$ (see [6]), then we obtain that, in $B_{\tau}(0) \setminus B_{\frac{5}{2}\delta}(y)$,

$$-\varepsilon^2 \Delta(\Psi_2(x) - \tilde{w}) + V(x)(\Psi_2(x) - \tilde{w}) \\ \geq -\varepsilon^2 \exp^{-\frac{c}{\varepsilon}} \sigma' \left(\log \frac{3|y-x|}{\delta} \right)^{-\sigma'-1} |y-x|^{-N} \left\{ N-2 + (\sigma'+1) \left(\log \frac{3|y-x|}{\delta} \right)^{-1} \right\} \\ + \exp^{-\frac{c}{\varepsilon}} \alpha \left(\log \frac{3|y-x|}{\delta} \right)^{-\beta-\sigma'} |y-x|^{-N} - w^p \geq 0,$$

and

$$|\tilde{w}(x)| \leq \Psi_2(x), \quad x \in \partial B_{\frac{5}{2}\delta}(y).$$

By the maximum principle, we obtain that $|\tilde{w}(x)| \leq \Psi_2(x)$ for any $x \in B_\tau(0) \setminus B_{\frac{\tau}{2}\delta}(y)$. Therefore, in the similar way of Proposition 2.2, we see that there exist $w_{\varepsilon,y,\tau} \in T_{\varepsilon,\tau}$ such that $I'_{\varepsilon,\tau}(U_{\varepsilon,y} + w_{\varepsilon,y,\tau}) = \sum_{i=1}^N C_{i,\varepsilon,y} \frac{\partial U_{\varepsilon,y}}{\partial x_i}$ for some $C_{i,\varepsilon,y} \in \mathbb{R}$. Moreover, in the same way of Lemma 2.3 and Proposition 2.4, we can solve the bifurcation equation.

Finally, by similar calculations as in proof of Theorem 1.1 for sufficiently small $\varepsilon > 0$, $p = \frac{N}{N-2}$ and some $y_M \in M^\delta$, (1.3) has a weak solution u_ε such that

$$u_\varepsilon(x) = U_{\varepsilon,y_M}(x) + w_{\varepsilon,y_M}(x),$$

where $\lim_{\varepsilon \rightarrow 0} \text{dist}(y_M, M) = 0$. Moreover, for given constant $\sigma' \geq \frac{N-2}{2}$, $\|w_{\varepsilon,y_M}\|_{L^\infty(B_{3\delta}(y_M))} \leq \varepsilon$ and

$$w_{\varepsilon,y_M}(x) \leq \exp^{-\frac{c}{\varepsilon}} |y_M - x|^{-N+2} \left(\log \frac{3|y_M - x|}{\delta} \right)^{-\sigma'}, \quad x \in \mathbb{R}^N \setminus B_{3\delta}(y_M). \quad \square$$

Remark 2.5. Using Lyapunov–Schmidt reduction method, we constructed solutions concentrating around some positive critical point of nonnegative V when (i) $N \geq 3$, $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ and V is not identically zero or (ii) $N \geq 3$, $p = \frac{N}{N-2}$ and $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 \log|x| > 0$. However, it is still an open question whether there exist bump solutions of (1.3) when (i) $N \geq 3$, $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$, $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 \log|x| = 0$ and $\limsup_{|x| \rightarrow \infty} V(x)|x|^2 \log|x| > 0$ or (ii) $N \geq 3$, $p = \frac{N}{N-2}$, $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 = 0$ and $\limsup_{|x| \rightarrow \infty} V(x)|x|^2 > 0$.

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