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TRANSITION LAYERS FOR A SPATIALLY INHOMOGENEOUS ALLEN-CAHN EQUATION IN MULTI-DIMENSIONAL DOMAINS

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Abstract. In this paper, we study a spatially inhomogeneous Allen-Cahn equation in multi-dimensional domains. By upper and lower solution method, we obtain a sufficient condition for a hypersurface S in the domain Ω to support stable transition layers, and a necessary condition for S in Ω to support transition layers, not necessarily stable. In addition, sharp estimates on depths of transition layers have also been derived.

1. **Introduction.**

Some classes of reaction-diffusion equations give rise to sharp transition layers when the diffusion coefficients are very small. Such phenomena have long been known in physics, biology and other areas of science. Since the middle of 1970s, they have become subjects of intensive mathematical studies.

The Allen-Cahn equation

$$\begin{cases} \epsilon u_t = \epsilon \Delta u + \frac{1}{\epsilon}(u - u^3) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

where Ω is a bounded smooth domain in R^N and ν is the unit outer normal vector to $\partial\Omega$, is a typical example in which sharp transition layers develop and evolve when ϵ is sufficiently small. There has been tremendous research dedicated to this problem and its stationary counterpart. See [1], [6], [7], [9], [17] and references therein.

From a viewpoint of application, it is natural to consider spatially inhomogeneous problems, since realistically speaking, environments where reaction and diffusion take place are not uniform. In this paper, we focus on the following spatially inhomogeneous Allen-Cahn equation

$$\begin{cases} \epsilon u_t = \epsilon \Delta u + \frac{1}{\epsilon} h^2(x) f(u) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{1}$$

where the coefficient $h(x) \in C^2(\Omega)$, a strictly positive and continuous function on $\overline{\Omega}$, represents spatial inhomogeneity of the media. The nonlinearity $f \in C^1$ satisfies the following conditions:

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(F1) f has precisely three zeros $\alpha^- < 0 < \alpha^+$ and satisfies

$$f'(\alpha^-) < 0,\ f'(0) > 0,\ f'(\alpha^+) < 0;$$

(F2) $\int_{\alpha^-}^{\alpha^+} f(u)du = 0$;

(F3) $\dfrac{f(u)}{u} > f'(u)$, for all $u \neq 0$.

Clearly, $f(u) = u - u^3$ meets with the above three conditions.

The aim of this paper is to study the qualitative properties related to stationary transition layered solutions in multi-dimensional domains. In one dimensional case, *very precise and complete* information concerning steady states with layers near maximum and/or minimum points of h have been obtained in [12], [13]. This paper can be viewed as the multi-dimensional counterpart of [13].

Let us formulate our problem more precisely. The stationary problem for (1) is written in the following form:

$$\begin{cases} \epsilon^2 \Delta u + h^2(x) f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The main approach employed in this paper is the construction of upper and/or lower solutions.

Definition 1.1. We say u is a weak upper (lower) solution of (2) if u is bounded and measurable in Ω and

$$\int_\Omega [u\epsilon^2 \Delta\varphi + \varphi h^2(x) f(u)]dx \leq (\geq) 0 \tag{3}$$

for all $\varphi \in C^2(\overline{\Omega})$ with $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ and $\varphi > 0$ in Ω.

For the problem (2), we first derive conditions on $h(x)$ which guarantee the existence of stable solutions of (2) with sharp transition near a given closed smooth hypersurfaces S in Ω.

Theorem 1.2. *Given a C^3 closed hypersurface S in Ω, suppose that on S, h satisfies the following two geometric hypotheses:*

(H1) $\dfrac{\partial h}{\partial \mu} - (N-1)\kappa h = 0$,

(H2) $\dfrac{\partial^2 h}{\partial \mu^2} > (N-1)^2 \kappa^2 h + h \displaystyle\sum_{i=1}^{N-1} \lambda_i^2$,

where μ is the unit inner normal vector to S, κ is the mean curvature of S and $\lambda_1, \lambda_2, ..., \lambda_{N-1}$ are the principal curvatures of S, then the equation (2) has at least two stable solutions with transition layers in an ϵ-neighborhood of S, for every sufficiently small $\epsilon > 0$.

Remark 1. The term "solutions with transition layers in an ϵ-neighborhood of S" will be explained after the statement of Theorem 2.1, which implies Theorem 1.2, in Section 2.2.

Notice that, if $N = 1$, the hypotheses (H1) and (H2) become $h' = 0$ and $h'' > 0$ at some point in Ω. These are exactly the hypotheses in [13] which imply the existence of stable steady states with transition layers in one dimensional case. This problem is also considered in [14] and [15]. In [15], for the radially symmetric case, the authors obtain a criterion which guarantees the existence of transition layered solutions, not necessarily stable, of (2).

When $\Omega = B_R$ ($B_{R_1} \setminus \overline{B_{R_2}}$, resp.) and $h(x) = h(r)$ where $R > 0$, $B_R = \{x \in R^N \mid |x| < R\}$ and $r = |x|$, it is proved in [8, Theorem 1.1] that any non-constant radially symmetric solution of (2) is unstable if

$$\left(r^{N-1}\left(\frac{1}{r^{N-1}h}\right)'\right)' \geq 0 \quad \text{in } B_R \; (B_{R_1} \setminus \overline{B_{R_2}}, \text{ resp.}). \tag{4}$$

However, from Theorem 1.2, when ϵ is sufficiently small, the equation (2) has stable radially symmetric solutions which have a transition layer in an ϵ-neighborhood of any given sphere $\{x \in R^N \mid |x| = r_0,\ 0 < r_0 < R\ (R_1 < r_0 < R_2, \text{ resp.})\}$ satisfying the hypotheses $(H1)$ and $(H2)$. By direct computation, the hypotheses $(H1)$ and $(H2)$ imply that

$$\left(r^{N-1}\left(\frac{1}{r^{N-1}h}\right)'\right)' < 0 \quad \text{at } r = r_0.$$

This is exactly the case in which (4) is violated.

It is worth pointing out that, in Theorem 1.2, for any closed C^3 hypersurface S in Ω, it is not difficult to show the existence of the strictly positive function $h \in C^2(\Omega)$ which satisfies the hypotheses (H1) and (H2) on S. Therefore, it is possible for the problem (2) to have stable solutions with layers appearing near any given C^3 closed hypersurface S in Ω as long as $h(x)$ is chosen properly. We include some simple examples of such $h(x)$ at the end of Section 2.1.

We ought to mention that, in [2] and [16], another type of inhomogeneous Allen-Cahn equation, which seems simpler compared to our problem, is considered.

Our next main result deals with a necessary condition for a closed hypersurface S in Ω to support transition layers.

Theorem 1.3. *Let S be a closed C^3 hypersurface in Ω. Suppose that there is a sequence of solutions u_{ϵ_n} of (2) with $\epsilon = \epsilon_n$ such that S is an isolated limit of the zero level sets of u_{ϵ_n}, as ϵ_n decreases to zero, then we have*

$$\frac{\partial h}{\partial \mu} - (N-1)\kappa h = 0 \quad \text{on } S.$$

Moreover, if

$$\frac{\partial^2 h}{\partial \mu^2} \neq (N-1)^2\kappa^2 h + h\sum_{i=1}^{N-1}\lambda_i^2$$

for every point on S, then the layers of the solution u_{ϵ_n} are in an $\epsilon_n|\log\epsilon_n|^2$-neighborhood of S, for n sufficiently large.

Remark 2. We say S in Ω is an isolated limit of the zero level sets of a sequence of solutions u_{ϵ_n} if there exist a neighborhood Ω_S of S in Ω and a sequence of neighborhoods Ω_n of S in Ω such that

$$\Omega_S \cap \{x \in \Omega \mid u_{\epsilon_n}(x) = 0\} \subset \Omega_n, \quad \bigcap_{n=1}^{\infty}\Omega_n = S.$$

The proof of Theorem 1.3 is based on the sharp estimates of decay rate of transition layers obtained in Proposition 1.

In fact, the condition imposed on S can be weakened, we will be more precise in Section 4. To the best of our knowledge, this is the first time in the literature such a necessary condition is derived.

Comparing Theorems 1.2 and 1.3, we suspect that the following condition

$$\frac{\partial^2 h}{\partial \mu^2} < (N-1)^2\kappa^2 h + h\sum_{i=1}^{N-1}\lambda_i^2$$

is related to the instability of transition layered solutions.

Our approach in this paper is both elementary and constructive. The proof itself automatically reveals the nature of those hypotheses required in the multi-dimensional case. We also remark that our main results in this paper concern sharp transition layers, thus we always assume that the diffusion coefficient ϵ is sufficiently small throughout the paper.

This paper is organized as follows. Theorem 1.2 is established in Section 2. Section 3 is devoted to the proof of Proposition 1. Finally, the proof of Theorem 1.3 is included in Section 4.

2. **Existence of stable transition layers.**

2.1. **Preliminaries.** W.l.o.g., let S be a connected C^3 closed hypersurface in Ω. We need to carry out our analysis near S; in particular, we need to compute the Laplacian in the following local coordinate system near S.

There exist $\{\gamma_i\}_{i=1}^m$,

$$\gamma_i:\ U_i\ \to\ V_i\cap S,$$

where U_i is an open set in R^{N-1} and V_i is an open set in R^N, with the following properties:

(i) γ_i is a C^3 differentiable homeomorphism;
(ii) $(d\gamma_i)_Q:\ R^{N-1}\to R^N$ is injective for all $Q\in U_i$;
(iii) $S\subset\bigcup_{i=1}^m\gamma_i(U_i)$.

It is clear that there exists $\delta>0$, sufficiently small, such that

$$S_\delta\equiv\left\{x\in R^N\mid dist(x,S)<\delta\right\}\subset\Omega \tag{5}$$

and the map

$$\sigma_i(s_1,...,s_{N-1},t)\equiv\gamma_i(s_1,...,s_{N-1})+t\mu[\gamma_i(s_1,...,s_{N-1})]$$

is a C^2 differentiable homeomorphism from $U_i\times(-\delta,\delta)$ to its image, for all $1\le i\le m$, where $\mu[\gamma_i(s_1,...,s_{N-1})]$ represents the unit inner normal vector to S at the point $\gamma_i(s_1,...,s_{N-1})$.

First of all, let us rewrite the Laplacian in the local coordinates s_1,...,s_{N-1}, t introduced above. For simplicity, we shall suppress the sub-index i in our computation below (when no confusion arises) and denote the transpose of the vector $x=(x_1,...,x_N)$ by x^T:

$$\gamma(s_1,...,s_{N-1})+t\mu[\gamma(s_1,...,s_{N-1})]=(x_1,...,x_N)^T.$$

(That is, we will be dealing with column vectors.)

Setting

$$\partial_N=\frac{\partial}{\partial t},\ \partial_j=\frac{\partial}{\partial s_j},\ j=1,...,N-1,$$

and

$$g_{jk}=<\partial_j,\partial_k>,\ 1\le j,k\le N,$$

$$G = (g_{jk})_{N\times N},\ G^{-1} = (g^{jk})_{N\times N},\ g = det\, G,$$

we obtain, from the standard computation, that

$$g_{jN} = 0 = g^{jN},\ 1 \le j \le N-1,\ g_{NN} = 1 = g^{NN}$$

and

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\sqrt{g} = -\sum_{i=1}^{N-1}\frac{\lambda_i}{1-t\lambda_i},$$

here $\lambda_1, ..., \lambda_{N-1}$ are the principal curvatures of S. Next if we define, on $U\times(-\delta,\delta)$,

$$\widetilde{u}(s_1, ..., s_{N-1}, t) = u(\sigma(s_1, ..., s_{N-1}, t)) = u(x_1, ..., x_N),$$

then

$$\begin{aligned}\Delta u &= \frac{1}{\sqrt{g}}\sum_{j,k=1}^{N}\partial_j\left(g^{jk}\sqrt{g}\partial_k\widetilde{u}\right)\\ &= \frac{1}{\sqrt{g}}\sum_{j,k=1}^{N-1}\partial_j\left(g^{jk}\sqrt{g}\partial_k\widetilde{u}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}\frac{\partial\widetilde{u}}{\partial t}\right).\end{aligned} \tag{6}$$

Our upper and lower solutions take the following form

$$u(x_1, ..., x_N) = v\left(\int_{t_0}^{t^i} h(\sigma_i(s_1^i, ..., s_{N-1}^i, \tau))d\tau\right) \tag{7}$$

for $x = (x_1, ..., x_N) = \sigma_i(s_1^i, ..., s_{N-1}^i, t^i) \in \sigma_i(U_i\times(-\delta,\delta))$, where $|t^i|,\ |t_0| < \delta$ (t_0 to be chosen), $v : R \to R$ is a suitable one-dimensional function, and h is the positive function in (2). To verify that the function given by (7) is well-defined in S_δ, we proceed as follows.

If

$$x = (x_1, ..., x_N) \in \sigma_i(U_i\times(-\delta,\delta)) \cap \sigma_j(U_j\times(-\delta,\delta))$$

for some $i \ne j$, then

$$\begin{aligned}(x_1, ..., x_N) &= \gamma_i(s_1^i, ..., s_{N-1}^i) + t^i\mu[\gamma_i(s_1^i, ..., s_{N-1}^i)]\\ &= \gamma_j(s_1^j, ..., s_{N-1}^j) + t^j\mu[\gamma_j(s_1^j, ..., s_{N-1}^j)].\end{aligned}$$

This implies that

$$\gamma_i(s_1^i, ..., s_{N-1}^i) = \gamma_j(s_1^j, ..., s_{N-1}^j)$$

is the unique point on S which is nearest to x, and $t^i = t^j =$ signed distance between x and S. Thus, for $\tau \in (-\delta,\delta)$,

$$\sigma_i(s_1^i, ..., s_{N-1}^i, \tau) = \sigma_j(s_1^j, ..., s_{N-1}^j, \tau)$$

and

$$\begin{aligned}\widetilde{h}(s_1^i, ..., s_{N-1}^i, \tau) &\equiv h(\sigma_i(s_1^i, ..., s_{N-1}^i, \tau)) = h(\sigma_j(s_1^j, ..., s_{N-1}^j, \tau))\\ &\equiv \widetilde{h}(s_1^j, ..., s_{N-1}^j, \tau).\end{aligned}$$

Hence the function u given by (7), in terms of v, is well-defined in S_δ.

Moreover, we remark that the hypothesis (H1) in the coordinates s_1,...,s_{N-1}, t takes the form

$$\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h} = 0 \quad \text{at } t = 0, \tag{8}$$

and (H2) becomes

$$\widetilde{h}_{tt} > (N-1)^2\kappa^2\widetilde{h} + \widetilde{h}\sum_{i=1}^{N-1}\lambda_i^2 \quad \text{at } t=0,$$

which, together with (H1), gives that

$$\frac{\partial}{\partial t}\left(\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h}\right) > 0 \quad \text{at } t=0. \tag{9}$$

Finally, let us explicitly write down some functions which satisfy the hypotheses (H1) and (H2) on a given closed C^3 hypersurface S. It is routine to verify that

$$h(x_1,...,x_N) = \widetilde{h}(s_1,...,s_{N-1},t) \equiv \exp\left\{(N-1)\kappa t + \frac{1}{2}\left(\sum_{i=1}^{N-1}\lambda_i^2 + c\right)t^2\right\},$$

where c is any given positive constant and $t \in (-\delta,\delta)$, satisfies the hypotheses (H1) and (H2) on S. And clearly we can always extend the above function h defined in S_δ to the whole domain Ω such that it is C^2 differentiable in Ω and positive on $\overline{\Omega}$.

From now on, we denote $s = (s_1,...,s_{N-1})$ when there is no confusion.

2.2. **Construction of Stable Transition Layers.** Our first main result here establishes the existence of a pair of stable transition layers of the problem (2), which includes Theorem 1.2, namely,

$$\begin{cases} \epsilon^2\Delta u + h^2(x)f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h > 0$ on $\overline{\Omega}$ and $f \in C^1$ satisfies (F1), (F2) and (F3): (in the Introduction)

(F1) f has precisely three zeros $\alpha^- < 0 < \alpha^+$ and satisfies

$$f'(\alpha^-) < 0,\ f'(0) > 0,\ f'(\alpha^+) < 0;$$

(F2) $\int_{\alpha^-}^{\alpha^+} f(u)du = 0$;

(F3) $\dfrac{f(u)}{u} > f'(u)$, for all $u \neq 0$.

Theorem 2.1. *Suppose that, on a C^3 closed hypersurface $S \subset \Omega$, the following hypotheses (H1) and (H2) hold:*

(H1) $\dfrac{\partial h}{\partial \mu} - (N-1)\kappa h = 0$,

(H2) $\dfrac{\partial^2 h}{\partial \mu^2} > (N-1)^2\kappa^2 h + h\displaystyle\sum_{i=1}^{N-1}\lambda_i^2$,

where μ is the unit inner normal vector to S, κ is the mean curvature of S and $\lambda_1,\lambda_2,...,\lambda_{N-1}$ are the principal curvatures of S. Then (2) has two stable solutions with transition layers in an ϵ-neighborhood of S, for every sufficiently small $\epsilon > 0$.

Remark 3. We say a nontrivial solution u_ϵ of (2) has a transition layer in an ϵ-neighborhood of S in Ω if for any $\sigma > 0$, there exists a positive constant C, independent of ϵ, such that when $d(x,S) > C\epsilon$, either $\alpha^- < u_\epsilon(x) < \alpha^- + \sigma$ or $\alpha^+ - \sigma < u_\epsilon(x) < \alpha^+$ holds in S_δ, where S_δ is specified in (5).

In order to define and verify our upper and lower solutions, the following three lemmas are necessary.

Lemma 2.2. *There exists a unique solution to the following problem*

$$\begin{cases} \phi_{zz}(z) + f(\phi(z)) = 0 & -\infty < z < \infty, \\ \lim_{z\to-\infty} \phi(z) = \alpha^-, \ \lim_{z\to\infty} \phi(z) = \alpha^+, \\ \phi(0) = 0. \end{cases} \tag{10}$$

Furthermore, $\phi_z(z) > 0$ *in* $(-\infty, \infty)$, $\lim\limits_{|z|\to\infty} \phi_z(z) = 0$ *and* $\dfrac{\phi_{zz}}{\phi_z}$ *is bounded in* $(-\infty, \infty)$.

This lemma is rather standard. We omit the proof.

Let R and $\widehat{R}$ be two positive numbers such that

$$0 < R < \min\left\{\sqrt{|f'(\alpha^+)|}, \ \sqrt{|f'(\alpha^-)|}\right\} \tag{11}$$

and

$$\widehat{R} > \max\left\{\sqrt{|f'(\alpha^+)|}, \ \sqrt{|f'(\alpha^-)|}\right\},$$

we define

$$v^+(z) \equiv \phi(z) + \beta\epsilon^2\theta(\frac{R}{2}z), \tag{12}$$

$$v^-(z) \equiv \phi(z) - \beta\epsilon^2\theta(\frac{R}{2}z), \tag{13}$$

where ϕ is the unique solution to the problem (10) and

$$\theta(z) \equiv \frac{z^2}{z^2+1}\exp(|z|) \quad \text{ for } -\infty < z < \infty. \tag{14}$$

The constant β here is to be specified later.

Now denote

$$\tau_1 = \min\left\{z > 0 \;\Big|\; \frac{d}{dz}v^+(-z) = 0\right\}, \ \tau_2 = \min\left\{z > 0 \mid v^+(z) = \alpha^+\right\},$$

$$\tau_3 = \min\left\{z > 0 \mid v^-(-z) = \alpha^-\right\}, \ \tau_4 = \min\left\{z > 0 \;\Big|\; \frac{d}{dz}v^-(z) = 0\right\}.$$

The following two lemmas give estimates for ϕ and $\{\tau_i\}_{i=1}^4$ respectively.

Lemma 2.3. *Let* $\phi(z)$ *be a unique solution of (10). There exist some positive constants* $0 < A_1 < A_2$ *such that the following inequalities hold*

$$A_1\exp(-\widehat{R}|z|) < \phi(z) - \alpha^- < A_2\exp(-R|z|), \ \ z \le 0,$$

$$A_1\exp(-\widehat{R}|z|) < \alpha^+ - \phi(z) < A_2\exp(-R|z|), \ \ z \ge 0,$$

$$A_1\exp(-\widehat{R}|z|) < \phi'(z) < A_2\exp(-R|z|), \ \ -\infty < z < \infty.$$

This lemma can be proved by following similar arguments in [4, Propostion 4.1]. We omit the proof here.

Lemma 2.4. $\{\tau_i\}_{i=1}^4$ *satisfy the following inequalities:*

$$\frac{3}{R+2\widehat{R}}|\log\epsilon| < \tau_i < \frac{5}{3R}|\log\epsilon|, \ i = 1, 2, 3, 4,$$

when ϵ *is sufficiently small.*

Proof. By the definition of τ_2, it satisfies

$$\phi(\tau_2) + \beta\epsilon^2\theta(\frac{R}{2}\tau_2) = \alpha^+.$$

Using (14) and Lemma 2.3, we get

$$\beta\epsilon^2 \exp\left(\frac{R}{2}\tau_2\right) \geq \alpha^+ - \phi(\tau_2) > A_1 \exp(-\widehat{R}\tau_2).$$

Consequently,

$$\tau_2 \geq \frac{3}{R + 2\widehat{R}} |\log \epsilon|$$

holds when ϵ is sufficiently small. The other inequality for τ_2 holds in the same way. By the same argument, we obtain the other inequalities. □

Next, we set

$$V^+(y) \equiv \begin{cases} v^+(-\tau_1) & y < -\epsilon\tau_1, \\ v^+(y/\epsilon) & -\epsilon\tau_1 \leq y \leq \epsilon\tau_2, \\ \alpha^+ & y > \epsilon\tau_2, \end{cases} \tag{15}$$

$$V^-(y) \equiv \begin{cases} \alpha^- & y < -\epsilon\tau_3, \\ v^-(y/\epsilon) & -\epsilon\tau_3 \leq y \leq \epsilon\tau_4, \\ v^-(\tau_4) & y > \epsilon\tau_4. \end{cases}$$

Finally, we are ready to define our two pairs of weak upper and lower solutions. For simplicity, we first denote the subdomain enclosed by S, by D; the subdomain of Ω complement to the region enclosed by the surface $\partial S_\delta \setminus D$, by Ω_{out}; and the subdomain enclosed by $\partial S_\delta \bigcap D$, by Ω_{in}, where S_δ is specified in (5). See Fig. 2.1.

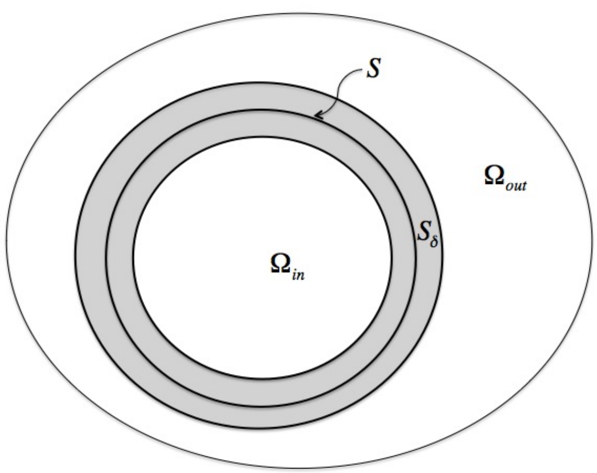

Fig. 2.1.

We define

$$U_L^+(x) \equiv \begin{cases} v^+(-\tau_1) & \text{if } x \in \overline{\Omega}_{out}, \\ V^+\left(\int_{-\epsilon\gamma}^{t} \widetilde{h}(s,\tau)d\tau\right) & \text{if } x = \sigma(s,t) \in S_\delta, \\ \alpha^+ & \text{if } x \in \overline{\Omega}_{in}, \end{cases} \tag{16}$$

$$U_L^-(x) \equiv \begin{cases} \alpha^- & \text{if } x \in \overline{\Omega}_{out}, \\ V^-\left(\int_{\epsilon\gamma}^t \widetilde{h}(s,\tau)d\tau\right) & \text{if } x = \sigma(s,t) \in S_\delta, \\ v^-(\tau_4) & \text{if } x \in \overline{\Omega}_{in}, \end{cases}$$

$$U_R^+(x) \equiv \begin{cases} \alpha^+ & \text{if } x \in \overline{\Omega}_{out}, \\ V^+\left(-\int_{\epsilon\gamma}^t \widetilde{h}(s,\tau)d\tau\right) & \text{if } x = \sigma(s,t) \in S_\delta, \\ v^+(-\tau_1) & \text{if } x \in \overline{\Omega}_{in}, \end{cases}$$

$$U_R^-(x) \equiv \begin{cases} v^-(\tau_4) & \text{if } x \in \overline{\Omega}_{out}, \\ V^-\left(-\int_{-\epsilon\gamma}^t \widetilde{h}(s,\tau)d\tau\right) & \text{if } x = \sigma(s,t) \in S_\delta, \\ \alpha^- & \text{if } x \in \overline{\Omega}_{in}, \end{cases}$$

where the constant γ is to be specified later. Clearly, U_L^+, U_L^-, U_R^+ and U_R^- are well-defined for every ϵ sufficiently small in view of our arguments in Section 2.1 following (7). As for the continuity of those functions, let us explain it by taking U_L^+ for example. Notice that, because of Lemma 2.4 and the fact that h is positive on $\overline{\Omega}$, $-\epsilon\tau_1 \le \int_{-\epsilon\gamma}^t \widetilde{h}(s,\tau)d\tau \le \epsilon\tau_2$ implies that t is in an $\epsilon|\log\epsilon|$-neighborhood of 0, which is contained in $(-\delta,\delta)$ if ϵ is small enough. Therefore, U_L^+ is continuous in Ω due to the definitions of functions in (15) and (16).

Lemma 2.5. *$U_L^+(x) > U_L^-(x)$ ($U_R^+(x) > U_R^-(x)$ resp.) in Ω.*

Proof. It follows immediately from the definitions of these functions. □

Let us specify β in (12) and γ in (16) as follows. According to the definition of θ in (14), straightforward calculation shows that

$$\theta''(z) - 2\theta(z) = \mathcal{T}(z)\exp(|z|),$$

where

$$\mathcal{T}(z) = -1 + \frac{z^4 + 4|z|^3 - 4z^2 + 4|z| + 3}{(z^2+1)^3}.$$

Obviously, there exists a constant $N_1 > 0$ independent of $\epsilon > 0$ such that

$$\theta''(z) - 2\theta(z) < 0 \quad \text{for } |z| > N_1.$$

Next recalling the definition of R in (11), it is obvious that there exists sufficiently large $N_2 > 0$ such that

$$-f'(\phi(z)) > \frac{3}{4}R^2 \quad \text{for } |z| > N_2.$$

Setting

$$N^* = \max\{N_1, N_2\}, \tag{17}$$

choose $\beta > 0$ sufficiently small and $\gamma > 0$ sufficiently large such that

$$\beta << \gamma\exp\left(-\left(\widehat{R} + \frac{R}{2}\right)N^*\right) \tag{18}$$

and

$$\gamma\exp\left(-\frac{3R}{2}\sqrt{\gamma}\right) << \beta. \tag{19}$$

Our main lemma is as follows.

Lemma 2.6. *Under conditions (8), (9), (18) and (19), U_L^+ and U_L^-, U_R^+ and U_R^- are two pairs of weak upper and lower solutions of the problem (2) respectively.*

Obviously, due to standard results (cf. e.g., [18, Theorem 3.6] and [11, Theorem 4]), Lemmas 2.5 and 2.6 immediately yield that (2) has two stable solutions u_1 and u_2, for sufficiently small $\epsilon > 0$. Moreover, because of the definitions of the functions U_L^+, U_L^-, U_R^+ and U_R^-, it is routine to check that the transition layers of these two stable solutions u_1 and u_2 are in an ϵ-neighborhood of S, for sufficiently small $\epsilon > 0$. Therefore, as long as Lemma 2.6 is verified, the proof of Theorem 2.1 is complete.

Proof of Lemma 2.6. We only show that U_L^+ is a weak upper solution of (2) since the other cases can be handled similarly.

For convenience we first denote the subdomain of Ω complement to the region enclosed by the surface

$$\widetilde{S}_1 \equiv \left\{ x = \sigma(s,t) \in S_\delta \;\middle|\; \int_{-\epsilon\gamma}^{t} \widetilde{h}(s,\tau)d\tau = -\epsilon\tau_1 \right\},$$

by Ω_1; the subdomain enclosed by the surface

$$\widetilde{S}_2 \equiv \left\{ x = \sigma(s,t) \in S_\delta \;\middle|\; \int_{-\epsilon\gamma}^{t} \widetilde{h}(s,\tau)d\tau = \epsilon\tau_2 \right\},$$

by Ω_2; and the subdomain

$$\left\{ x = \sigma(s,t) \in S_\delta \;\middle|\; -\epsilon\tau_1 < \int_{-\epsilon\gamma}^{t} \widetilde{h}(s,\tau)d\tau < \epsilon\tau_2 \right\}$$

by Ω_0. Fig. 2.2 shows the graph of U_L^+ along t-axis.

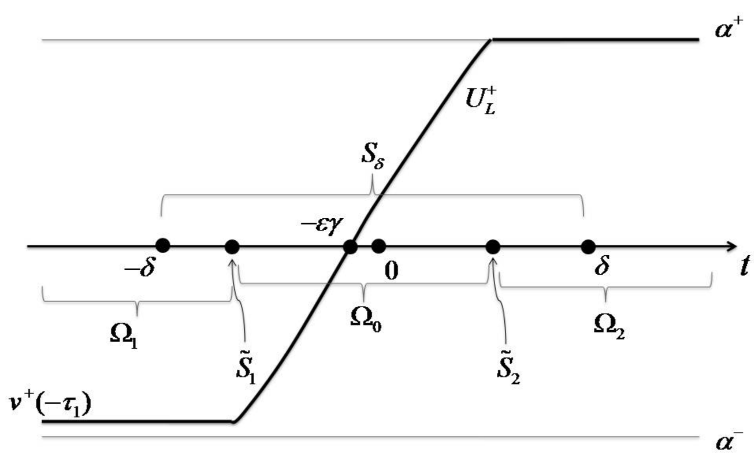

Fig. 2.2.

We claim that *under conditions (8), (9), (18) and (19),*

$$\epsilon^2 \Delta U_L^+ + h^2(x) f(U_L^+) \le 0 \quad \text{in } \Omega_0 \tag{20}$$

for all $\epsilon > 0$ sufficiently small.

Since the proof of this claim is lengthy, we prefer to finish the rest of the proof first based on the inequality (20). Obviously, U_L^+ is bounded and measurable in Ω. Notice that $f(U_L^+) \equiv f(v^+(-\tau_1)) < 0$ in Ω_1, $f(U_L^+) \equiv f(\alpha^+) = 0$ in Ω_2 and U_L^+ is continuous in Ω. Therefore for any $\varphi \in C^2(\overline{\Omega})$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ and $\varphi > 0$ in Ω, the divergence theorem implies that

$$\begin{aligned} & \int_\Omega \left[U_L^+ \epsilon^2 \Delta\varphi + \varphi h^2(x) f(U_L^+)\right] dx \\ < \; & \int_{\Omega_1} U_L^+ \epsilon^2 \Delta\varphi dx + \int_{\Omega_0} \left[U_L^+ \epsilon^2 \Delta\varphi + \varphi h^2(x) f(U_L^+)\right] dx + \int_{\Omega_2} U_L^+ \epsilon^2 \Delta\varphi dx \\ = \; & \int_{\Omega_0} \left[-\epsilon^2 \nabla U_L^+ \nabla\varphi + \varphi h^2(x) f(U_L^+)\right] dx. \end{aligned}$$

Applying the divergence theorem again, we have

$$\begin{aligned} & \int_{\Omega_0} \left[-\epsilon^2 \nabla U_L^+ \nabla\varphi + \varphi h^2(x) f(U_L^+)\right] dx \\ = \; & \int_{\Omega_0} \varphi \left[\epsilon^2 \Delta U_L^+ + h^2(x) f(U_L^+)\right] dx + \int_{\widetilde{S}_1} \epsilon^2 \varphi \frac{\partial U_L^+}{\partial \nu_1} + \int_{\widetilde{S}_2} \epsilon^2 \varphi \frac{\partial U_L^+}{\partial \nu_2}, \end{aligned}$$

where ν_1, ν_2 are the unit outer normal on $\widetilde{S}_1$, $\widetilde{S}_2$ to Ω_1, Ω_2, respectively. It follows from the definition of the function U_L^+ in (16) that

$$\frac{\partial U_L^+}{\partial \nu_1} = V_y^+(-\epsilon\tau_1) \left[\nabla_x \int_{-\epsilon\gamma}^t \widetilde{h}(s,\tau) d\tau\right] \cdot \nu_1 = 0 \quad \text{on } \widetilde{S}_1,$$

$$\frac{\partial U_L^+}{\partial \nu_2} = -\lim_{y \to \epsilon\tau_2^-} V_y^+(y) \left|\nabla_x \int_{-\epsilon\gamma}^t \widetilde{h}(s,\tau) d\tau\right| \leq 0 \quad \text{on } \widetilde{S}_2.$$

Therefore, combined with (20), we have

$$\int_\Omega \left[U_L^+ \epsilon^2 \Delta\varphi + \varphi h^2(x) f(U_L^+)\right] dx \leq 0,$$

and thus U_L^+ is a weak upper solution of (2).

Now let us verify (20). All the positive constants mentioned in the following are independent of $\epsilon > 0$. Set

$$y = \int_{-\epsilon\gamma}^t \widetilde{h}(s,\tau) d\tau \text{ and } z = \frac{y}{\epsilon}.$$

Clearly, $-\tau_1 < z < \tau_2$ because of the definition of Ω_0. We only show our proof when $0 \leq z < \tau_2$, that is $0 \leq \int_{-\epsilon\gamma}^t \widetilde{h}(s,\tau) d\tau < \varepsilon\tau_2$. It can proved similarly when $-\tau_1 < z \leq 0$. From (6), we deduce

$$\begin{aligned} & \frac{1}{h^2} \left[\epsilon^2 \Delta U_L^+ + h^2 f(U_L^+)\right] \\ = \; & \epsilon^2 \frac{1}{\widetilde{h}^2} \frac{1}{\sqrt{g}} \sum_{j,k=1}^{N-1} \partial_j \left(g^{jk} \sqrt{g} \partial_k \widetilde{U}_L^+\right) + \epsilon^2 \frac{1}{\widetilde{h}^2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left(\sqrt{g} \frac{\partial}{\partial t} \widetilde{U}_L^+\right) + f(\widetilde{U}_L^+) \\ = \; & I + J. \end{aligned}$$

Here

$$\begin{aligned} I &\equiv \epsilon^2 \frac{1}{\widetilde{h}^2} \sum_{j,k=1}^{N-1} g^{jk} \left(V^+_{yy} - \frac{1}{\epsilon}\frac{\phi_{zz}}{\phi_z} V^+_y \right) \int_{-\epsilon\gamma}^t \widetilde{h}_{s_j} d\tau \int_{-\epsilon\gamma}^t \widetilde{h}_{s_k} d\tau \\ &= \frac{1}{\widetilde{h}^2} \sum_{j,k=1}^{N-1} g^{jk} \left(v^+_{zz} - \frac{\phi_{zz}}{\phi_z} v^+_z \right) \int_{-\epsilon\gamma}^t \widetilde{h}_{s_j} d\tau \int_{-\epsilon\gamma}^t \widetilde{h}_{s_k} d\tau, \\ J &\equiv \epsilon^2 V^+_{yy} + \epsilon^2 m(s,t) V^+_y + f(V^+) \\ &= v^+_{zz} + \epsilon m(s,t) v^+_z + f(v^+), \end{aligned}$$

where

$$\begin{aligned} m(s,t) \equiv \frac{1}{\widetilde{h}^2} \Bigg\{ & \widetilde{h}_t + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial t} \widetilde{h} \\ & + \frac{1}{\epsilon} \sum_{j,k=1}^{N-1} g^{jk} \frac{\phi_{zz}}{\phi_z} \int_{-\epsilon\gamma}^t \widetilde{h}_{s_j} d\tau \int_{-\epsilon\gamma}^t \widetilde{h}_{s_k} d\tau \\ & + \frac{1}{\sqrt{g}} \sum_{j,k=1}^{N-1} \left[\partial_j (g^{jk}\sqrt{g}) \int_{-\epsilon\gamma}^t \widetilde{h}_{s_k} d\tau + g^{jk}\sqrt{g} \int_{-\epsilon\gamma}^t \widetilde{h}_{s_j s_k} d\tau \right] \Bigg\}. \end{aligned}$$

By Lemma 2.2 and the definition of v^+ in (12), direct calculation yields that

$$\left| v^+_{zz} - \frac{\phi_{zz}}{\phi_z} v^+_z \right| \le c_1' \beta \epsilon^2 \left(\left| \theta''(\frac{R}{2}z) \right| + \left| \theta'(\frac{R}{2}z) \right| \right),$$

for some positive constant c_1'. Hence, combined with Lemma 2.4, we have

$$\begin{aligned} & \frac{1}{h^2} \left[\epsilon^2 \Delta U_L^+ + h^2 f(U_L^+) \right] \\ \le \; & \epsilon m(s,t) \left[\phi'(z) + \frac{R}{2}\beta\epsilon^2 \theta'(\frac{R}{2}z) \right] + \beta\epsilon^2 \left[\frac{R^2}{4} \theta''(\frac{R}{2}z) + f'(\xi(z))\theta(\frac{R}{2}z) \right] \\ & + c_1 \beta \epsilon^4 |\log\epsilon|^2 \left(\left| \theta''(\frac{R}{2}z) \right| + \left| \theta'(\frac{R}{2}z) \right| \right), \end{aligned} \tag{21}$$

for some positive constant c_1, where

$$\phi(z) \le \xi(z) \le \phi(z) + \beta\epsilon^2 \theta(\frac{R}{2}z) \le \alpha^+. \tag{22}$$

We claim that *there exists* $\rho(\gamma) \in [c_2\sqrt{\gamma}, c_2'\sqrt{\gamma}]$ *such that*

$$m(s,t) \le 0 \quad \text{for } 0 \le z \le \rho(\gamma), \tag{23}$$

where c_2, c_2' $(c_2 < c_2')$ *are positive constants independent of* γ *and* ϵ.

By (8) and (9), there exists $\alpha > 0$ such that

$$\frac{\partial}{\partial t} \left[\frac{1}{\widetilde{h}^2} \left(\widetilde{h}_t + \frac{1}{\sqrt{g}} \frac{\partial\sqrt{g}}{\partial t} \widetilde{h} \right) \right] > \alpha \quad \text{at } t = 0.$$

Then it is easy to check that for $-\epsilon\gamma < t < 0$

$$m(s,t) \le \alpha t + \frac{1}{\epsilon} c_3 (t+\epsilon\gamma)^2 + c_3(t+\epsilon\gamma), \tag{24}$$

where c_3 is a positive constant satisfying

$$(N-1)^2 M_1 M_2^2 M_4 < c_3 h^2_{\min} \text{ and } (N-1)^2 (M_2 M_3 + M_4 M_5) < c_3 h^2_{\min}$$

with

$$\begin{aligned}
h_{\min} &= \min\left\{h(x) \mid x\in\overline{\Omega}\right\},\\
M_1 &= \max\left\{\left|\frac{\phi_{zz}}{\phi_z}\right| \,\Big|\, -\infty<z<\infty\right\},\\
M_2 &= \max\left\{\,|\widetilde{h}_{s_j}(s,t)| \,\big|\, \sigma(s,t)\in S_\delta, 1\le j\le N-1\,\right\},\\
M_3 &= \max\left\{\,\left|\frac{1}{\sqrt{g}}\partial_j(g^{jk}\sqrt{g})\right| \,\Big|\, \sigma(s,t)\in S_\delta, 1\le j,k\le N-1\,\right\},\\
M_4 &= \max\left\{\,|g^{jk}| \,\big|\, \sigma(s,t)\in S_\delta, 1\le j,k\le N-1\,\right\},\\
M_5 &= \max\left\{\,|\widetilde{h}_{s_js_k}(s,t)| \,\big|\, \sigma(s,t)\in S_\delta, 1\le j,k\le N-1\,\right\}.
\end{aligned}$$

Here $M_1<\infty$ by Lemma 2.2 and obviously M_i, $2\le i\le 5$, are all bounded. From (24), if $-\epsilon\gamma\le t\le -\epsilon\gamma+l\sqrt{\gamma}\epsilon$, where $l>0$ is small enough and independent of γ and ϵ, we have $m(s,t)\le 0$. This means that $m(s,t)\le 0$ holds if

$$z=\frac{1}{\epsilon}\int_{-\epsilon\gamma}^{t}\widetilde{h}(s,\tau)d\tau\le\frac{1}{\epsilon}\int_{-\epsilon\gamma}^{-\epsilon\gamma+l\sqrt{\gamma}\epsilon}\widetilde{h}(s,\tau)d\tau.$$

Setting

$$\begin{aligned}
\rho(\gamma) &= \frac{1}{\epsilon}\int_{-\epsilon\gamma}^{-\epsilon\gamma+l\sqrt{\gamma}\epsilon}\widetilde{h}(s,\tau)d\tau,\\
c_2 &= l\min\left\{\,|\widetilde{h}(s,t)| \,\big|\, \sigma(s,t)\in S_\delta\,\right\},\\
c_2' &= l\max\left\{\,|\widetilde{h}(s,t)| \,\big|\, \sigma(s,t)\in S_\delta\,\right\},
\end{aligned}$$

it follows that $m(s,t)\le 0$ when $0\le z\le\rho(\gamma)\in[c_2\sqrt{\gamma},c_2'\sqrt{\gamma}]$. The claim is proved.

Now let us go back to (21) and continue to show that

$$\frac{1}{h^2}\left[\epsilon^2\Delta U_L^+ + h^2 f(U_L^+)\right]\le 0.$$

We will handle the following three cases separately.

Case 1: $0\le z\le N^*$, where N^* is from (17). On this interval, there exists some constant $c_4>0$ such that

$$m(s,t)\le -c_4\gamma\epsilon \tag{25}$$

and

$$\left|\frac{R^2}{4}\theta''(\frac{R}{2}z)+f'(\xi(z))\theta(\frac{R}{2}z)\right|\le c_4\exp\left(\frac{R}{2}N^*\right). \tag{26}$$

Using (21), (25), (26) and Lemma 2.3, we have

$$\frac{1}{h^2}\left[\epsilon^2\Delta U_L^+ + h^2 f(U_L^+)\right]\le -A_1c_4\gamma\epsilon^2\exp(-\widehat{R}N^*)+\beta c_4\epsilon^2\exp\left(\frac{R}{2}N^*\right)+O(\epsilon^3).$$

Then (18) yields that the right hand side of the above inequality is negative if ϵ is sufficiently small.

Case 2: $N^*\le z\le\rho(\gamma)$. On this interval $m(s,t)\le 0$ from (23) and $\phi'(z)\ge 0$. Moreover, it follows from (17) and (22) that

$$\frac{R^2}{4}\theta''(\frac{R}{2}z)+f'(\xi(z))\theta(\frac{R}{2}z)<-\frac{R^2}{4}\theta(\frac{R}{2}z). \tag{27}$$

Hence we conclude that

$$\frac{1}{h^2}\left[\epsilon^2 \Delta U_L^+ + h^2 f(U_L^+)\right] \leq -\frac{R^2}{4}\beta\epsilon^2\theta(\frac{R}{2}z) + O(\epsilon^3) \leq 0,$$

if ϵ is sufficiently small.

Case 3: $\rho(\gamma) \leq z \leq \tau_2$. On this interval, it holds that

$$|m(s,t)| \leq c_5\epsilon z^2, \quad \phi'(z) \leq A_2 \exp(-Rz),$$

where c_5 is a positive constant. Notice that (27) also holds on this interval. Therefore, together with (19), we have

$$\begin{aligned}
& \frac{1}{h^2}\left[\epsilon^2 \Delta U_L^+ + h^2 f(U_L^+)\right] \\
\leq \quad & c_5 A_2 \epsilon^2 z^2 \exp(-Rz) - \frac{R^2}{4}\beta\epsilon^2\theta(\frac{R}{2}z) + O(\epsilon^3) \\
\leq \quad & c_5 A_2 \epsilon^2 c_2^2 \gamma \exp(-Rc_2\sqrt{\gamma}) - \frac{R^2}{4}\beta\epsilon^2\theta(\frac{R}{2}c_2\sqrt{\gamma}) + O(\epsilon^3) \leq 0,
\end{aligned}$$

if ϵ is sufficiently small.

The proof is complete. □

3. **Depth of transition layers.** In this section, we will provide the estimates concerning the depth of the transition layers. Let u_ϵ be a nontrivial solution of the problem (2) as follows

$$\begin{cases} \epsilon^2 \Delta u + h^2(x) f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

and Z_ϵ be the set of zeros of u_ϵ. Set $d(x) = dist(x, Z_\epsilon)$. Our estimates are stated as follows.

Proposition 1. *There exist $\eta_1 > 0$, $C_1 > 0$, independent of ϵ, such that either of the following inequalities holds*

$$0 < \alpha^+ - u_\epsilon(x) < C_1 \exp\left(-\frac{\eta_1 d(x)}{\epsilon}\right), \tag{28}$$

$$0 < u_\epsilon(x) - \alpha^- < C_1 \exp\left(-\frac{\eta_1 d(x)}{\epsilon}\right), \tag{29}$$

for ϵ sufficiently small.

Remark 4. We want to point out that these estimates hold for any non-trivial solutions of the problem (2). Moreover, in this paper, these estimates play an important role in the proof of Theorem 1.3.

The proof of Proposition 1 is lengthy. To better explain our idea, we present the main steps of the proof in a series of lemmas.

Lemma 3.1. *There exists $M_0 > 0$ such that for $M \geq M_0$, the following problem has the unique positive solution w_M:*

$$\begin{cases} \Delta w + f(w) = 0 & \text{in } B_M, \\ w = 0 & \text{on } \partial B_M. \end{cases} \tag{30}$$

Moreover, w_M is radially symmetric and $0 \leq \alpha^+ - w_M(0) \leq cM^{-2}$, where $M \geq M_0$ and c is a positive constant independent of M.

Proof. Obviously, there exists $M_0 > 0$ such that the constant solution $w = 0$ of (30) is unstable if $M \geq M_0$. In this case it is well known (see [10, Theorem 3.1], for example) that (30) has a positive solution. Because f satisfies (F3) in the Introduction, we can prove the uniqueness of the positive solution to (30). We denote this solution by w_M. The solution w_M is radially symmetric by the Gidas-Ni-Nirenberg symmetry result [3].

Now, suppose that the set $\{x \in B_M \mid w_M > \alpha^+\}$ is not empty. By applying maximum principle in any connected component of this set, it is easy to derive a contradiction. Hence, $\alpha^+ - w_M(0) \geq 0$.

Next, let μ_1 denote the first eigenvalue of

$$\begin{cases} \Delta\psi + \mu\psi = 0 & \text{in } B_1, \\ \psi = 0 & \text{on } \partial B_1. \end{cases}$$

Then the first eigenvalue of

$$\begin{cases} \Delta\varphi + \mu\varphi = 0 & \text{in } B_M, \\ \varphi = 0 & \text{on } \partial B_M. \end{cases}$$

is $\mu_1 M^{-2}$. Denote the first eigenfunction by φ_1. The fact that φ_1 is radially symmetric follows from the simplicity of the first eigenvalue. Moreover, because of the conditions imposed on f, there exists $\sigma > 0$, such that

$$\frac{f(w)}{w} \geq \sigma(\alpha^+ - w) \quad \text{for } 0 \leq w \leq \alpha^+. \tag{31}$$

In fact we will prove that for $M \geq M_0$

$$\alpha^+ - w_M(0) \leq \mu_1 M^{-2}/\sigma.$$

Suppose that there exists some $\mathcal{M} > M_0$ such that

$$\alpha^+ - w_{\mathcal{M}}(0) > \mu_1 \mathcal{M}^{-2}/\sigma.$$

Then from (30) and (31) we have

$$\Delta w_{\mathcal{M}} + \mu_1 \mathcal{M}^{-2} w_{\mathcal{M}} \leq 0.$$

Hence

$$\Delta(l\varphi_1 - w_{\mathcal{M}}) + \mu_1 \mathcal{M}^{-2}(l\varphi_1 - w_{\mathcal{M}}) \geq 0 \quad \text{for any } l > 0.$$

Obviously, when l is sufficiently small, $l\varphi_1 < w_{\mathcal{M}}$ in $B_{\mathcal{M}}$. As l increases, there are only two possibilities because $w_{\mathcal{M}}$ is bounded. The first one is that there exists some $\widetilde{l}$ such that $\widetilde{l}\varphi_1$ touches $w_{\mathcal{M}}$ at some points in $B_{\mathcal{M}}$ and $\widetilde{l}\varphi_1 \leq w_{\mathcal{M}}$ in $B_{\mathcal{M}}$. This is a contradiction to strong maximum principle [5, Page 35]. The second one is that there exists some $\widehat{l}$ such that $\widehat{l}\varphi_1'(\mathcal{M}) = w_{\mathcal{M}}'(\mathcal{M})$. This contradicts to Hopf boundary lemma [5, Page 34]. The proof is complete. □

Let us prepare some notations for convenience and then continue the proof of Proposition 1. Set

$$B(a;\ r) = \left\{x \in \Omega \;\middle|\; |x - a| < r\right\}, \quad h_{\min} = \min\left\{h(x) \mid x \in \overline{\Omega}\right\}.$$

By the properties of f, we can choose the constants H close to $-f'(\alpha^+)$ and $A > 0$ large enough such that

$$-f'(s) \geq H > 0 \quad \text{for } \alpha^+ - 2c(h_{\min}A)^{-2} \leq s < \alpha^+, \tag{32}$$

and

$$h_{\min}A > M_0, \tag{33}$$

where c and M_0 are the constants appearing in Lemma 3.1. For any $x_0 \in \Omega$ satisfying $d(x_0) \le 3\epsilon A$, (28) or (29) holds if $C_1 > 0$ is large enough. Now it remains to establish the proposition for $x_0 \in \Omega$ satisfying $d(x_0) > 3\epsilon A$.

It is clear from the definition of $d(x)$ that either $u_\epsilon > 0$ or $u_\epsilon < 0$ holds in $B(x_0;\ d(x_0))$. W.l.o.g., we assume that $u_\epsilon > 0$ in $B(x_0;\ d(x_0))$. With the help of the previous lemma, we have the following preliminary estimate.

Lemma 3.2 (See Fig. 3.1). *For any fixed* $x_1 \in \partial B(x_0;\ \frac{1}{3}d(x_0))$,

$$u_\epsilon(a) \ge \alpha^+ - c(h_{\min}A)^{-2} \quad \text{for } a \in \partial B(x_1;\ \frac{2}{3}d(x_0) - \epsilon A). \tag{34}$$

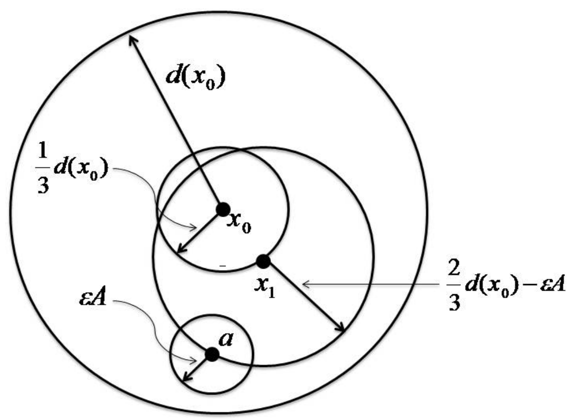

Fig. 3.1. $x_0 \in \Omega$, $d(x_0) > 3\varepsilon A$, $u_\varepsilon > 0$ in $B(x_0;\ d(x_0))$

Proof. It is obvious that $u_\epsilon > 0$ in $B(a;\ \epsilon A)$. To estimate u_ϵ, we first consider

$$\begin{cases} \epsilon^2 \Delta v + h_{\min}^2 f(v) = 0 & \text{in } B(a;\ \epsilon A), \\ v = 0 & \text{on } \partial B(a;\ \epsilon A). \end{cases} \tag{35}$$

Then

$$w(z) = v\left(a + \frac{\epsilon}{h_{\min}} z\right)$$

solves

$$\begin{cases} \Delta w + f(w) = 0 & \text{in } B(0;\ h_{\min}A), \\ w = 0 & \text{on } \partial B(0;\ h_{\min}A). \end{cases} \tag{36}$$

By Lemma 3.1 and (33), (36) has a solution w_1 satisfying

$$0 \le \alpha^+ - w_1(0) \le c(h_{\min}A)^{-2},$$

which directly imples that (35) has a solution v_1 and

$$0 \le \alpha^+ - v_1(a) \le c(h_{\min}A)^{-2}. \tag{37}$$

Notice that v_1 satisfies

$$\begin{cases} \epsilon^2 \Delta v + h(x)^2 f(v) \ge 0 & \text{in } B(a;\ \epsilon A), \\ v = 0 & \text{on } \partial B(a;\ \epsilon A). \end{cases} \tag{38}$$

Then it is easy to check that, for any $0 \le l \le 1$, lv_1 also satisfies (38) because of the property (F3) of f (in the Introduction). Suppose that there exists $x \in B(a;\ \epsilon A)$ such that $v_1(x) > u_\epsilon(x)$. Observe that when l is close to 0, $lv_1(x) < u_\epsilon(x)$ for $x \in B(a;\ \epsilon A)$, since $u_\epsilon > 0$ in $B(x_0;\ d(x_0)) \supset B(a;\ \epsilon A)$. Increasing l, a contradiction can be derived by the same argument at the end of the proof of Lemma 3.1. Hence $v_1(x) \le u_\epsilon(x)$ for $x \in B(a;\ \epsilon A)$. Combined with (37), we obtain the estimate (34). The proof is complete. □

Next, for any fixed $x_1 \in \partial B(x_0;\ \frac{1}{3}d(x_0))$, consider

$$\begin{cases} \epsilon^2 \Delta v + h_{\min}^2 f(v) = 0 & \text{in } B(x_1;\ \frac{2}{3}d(x_0) - \epsilon A), \\ v = \alpha^+ - 2c(h_{\min}A)^{-2} & \text{on } \partial B(x_1;\ \frac{2}{3}d(x_0) - \epsilon A). \end{cases} \tag{39}$$

Define

$$w(z) = v\left(x_1 + \frac{\epsilon}{h_{\min}} z\right)$$

which solves

$$\begin{cases} \Delta w + f(w) = 0 & \text{in } B(0;\ M_\epsilon), \\ w = \alpha^+ - b & \text{on } \partial B(0; M_\epsilon)\ , \end{cases} \tag{40}$$

where

$$b = \frac{2c}{(h_{\min}A)^2}, \quad M_\epsilon = h_{\min}\left(\frac{2}{3}\frac{d(x_0)}{\epsilon} - A\right).$$

The key ingredient in our proof is the following lemma.

Lemma 3.3. *There exists a solution w_2 to (40).*

Remark 5. This lemma is proved by constructing a pair of upper and lower solutions. We want to point out that the lower solution constructed in the proof below is crucial. Actually, because of the relation between the problems (39) and (40), our lower solution provides the lower bound to the solution of the problem (39). While, at the same time, thanks to Lemma 3.2, the solution of the problem (39) is comparable to u_ϵ, the solution of the problem (2). And Proposition 1 follows. We will strictly demonstrate these arguments after the proof of Lemma 3.3.

Proof of Lemma 3.3. Set

$$w(r) = \alpha^+ - b\exp(-\eta(M_\epsilon - r)) - b\gamma g(r)\left(\frac{M_\epsilon}{2} - r\right)^2 \exp\left(-\eta\frac{M_\epsilon}{2}\right), \tag{41}$$

where ϵ, η and γ are sufficiently small such that

$$M_\epsilon > \frac{2\eta}{\gamma}, \quad H > \eta^2 + \frac{4(N-1)}{M_\epsilon}\eta, \tag{42}$$

$$\gamma < \frac{H - \eta^2}{2}\exp\left(-\eta\sqrt{\frac{2}{H}}\right), \quad M_\epsilon > \frac{4(N-1)\eta}{H - \eta^2 - 2\gamma\exp\left(\eta\sqrt{\frac{2}{H}}\right)}, \tag{43}$$

where H comes from (32), and

$$g(r) \equiv \begin{cases} 1 & \text{for } r \le \frac{M_\epsilon}{2}, \\ 0 & \text{for } r > \frac{M_\epsilon}{2}. \end{cases}$$

About the function $w(r)$ defined in (41), direct calculations yield that, for $\frac{M_\epsilon}{2} < r \le M_\epsilon$, $w'(r) = -b\eta\exp(-\eta(M_\epsilon - r)) < 0$, and for $0 \le r \le \frac{M_\epsilon}{2}$,

$$w'(r) = -b\eta\exp(-\eta(M_\epsilon - r)) + 2b\gamma\left(\frac{M_\epsilon}{2} - r\right)\exp\left(-\eta\frac{M_\epsilon}{2}\right),$$

$$w''(r) = -b\eta^2\exp(-\eta(M_\epsilon - r)) - 2b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right) < 0.$$

Moreover, it is routine to check that $w'(\frac{M_\epsilon}{4}) > 0$ and $w'(\frac{M_\epsilon}{2}) < 0$ due to the condition $M_\epsilon > \frac{2\eta}{\gamma}$ in (42). Therefore, there exists $l^* \in (\frac{M_\epsilon}{4}, \frac{M_\epsilon}{2})$ such that $w'(l^*) = 0$. In fact, from previous calculations, it is obvious that $r = l^*$ is the only critical number of $w(r)$ in $(0, M_\epsilon)$. See Fig. 3.2.

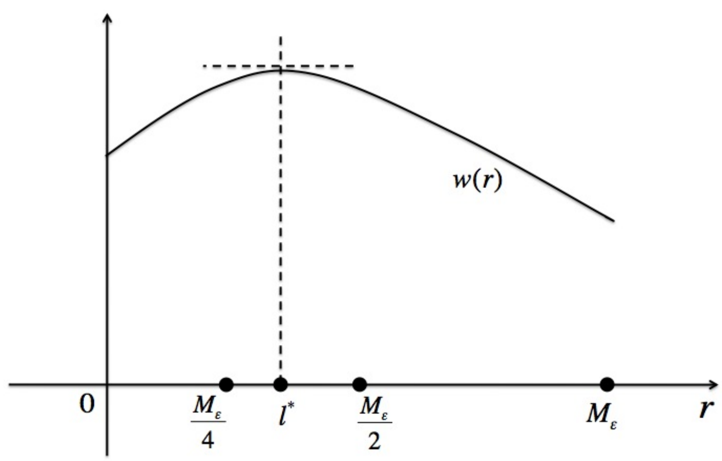

Fig. 3.2. Graph of $w(r)$

Now let us define

$$w^-(z) \equiv \begin{cases} w(l^*) & \text{for } |z| \le l^*, \\ w(|z|) & \text{for } l^* < |z| \le M_\epsilon. \end{cases} \tag{44}$$

and then show that $w^-(z)$ is a lower solution to (40).

For $|z| \le l^*$, it follows from the definition of w and (32) that

$$\Delta w^- + f(w^-) = f(w(l^*)) > f(\alpha^+) = 0.$$

For $l^* < r \le M_\epsilon$, $r \ne M_\epsilon/2$, where $r = |z|$. Here, the point $r = M_\epsilon/2$ is excluded since $g(r)$ is discontinuous at $r = M_\epsilon/2$. First, direct calculations give that

$$\begin{aligned} & \Delta w^- + f(w^-) \\ = \;& w_{rr} + \frac{N-1}{r} w_r + f(w) \\ = \;& -b\eta^2\exp(-\eta(M_\epsilon - r)) - 2g(r)b\gamma\exp(-\eta\frac{M_\epsilon}{2}) \\ & + \frac{N-1}{r}\left[-b\eta\exp(-\eta(M_\epsilon - r)) + 2g(r)b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right)\left(\frac{M_\epsilon}{2} - r\right)\right] \\ & + f'(\xi(r))\left[-b\exp(-\eta(M_\epsilon - r)) - g(r)b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right)\left(\frac{M_\epsilon}{2} - r\right)^2\right], \end{aligned}$$

where $w(r) \le \xi(r) \le \alpha^+$. Next, by the definition of $g(r)$, $l^* \in (\frac{M_\epsilon}{4}, \frac{M_\epsilon}{2})$ and (32), we derive that

$$\begin{aligned}
& \Delta w^- + f(w^-) \\
\ge \;& - b\eta^2 \exp(-\eta(M_\epsilon - r)) - 2g(r)b\gamma \exp\left(-\eta\frac{M_\epsilon}{2}\right) \\
& + \frac{N-1}{r}\left[-b\eta\exp(-\eta(M_\epsilon - r)) + 2g(r)b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right)\left(\frac{M_\epsilon}{2} - r\right)\right] \\
& + H\left[b\exp(-\eta(M_\epsilon - r)) + g(r)b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right)\left(\frac{M_\epsilon}{2} - r\right)^2\right] \\
\ge \;& \left(-\eta^2 - \frac{4(N-1)}{M_\epsilon}\eta + H\right) b\exp(-\eta(M_\epsilon - r)) \\
& + \left[-2 + H(\frac{M_\epsilon}{2} - r)^2\right] g(r)b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right).
\end{aligned} \tag{45}$$

Then we handle the following three cases separately.

Case 1: $l^* \le r \le \frac{M_\epsilon}{2} - \sqrt{\frac{2}{H}}$. $\Delta w^- + f(w^-) \ge 0$ follows directly from (42) and (45).

Case 2: $\frac{M_\epsilon}{2} - \sqrt{\frac{2}{H}} \le r \le \frac{M_\epsilon}{2}$. By (42), (43) and (45), we have

$$\begin{aligned}
& \Delta w^- + f(w^-) \\
\ge \;& \left(-\eta^2 - \frac{4(N-1)}{M_\epsilon}\eta + H\right) b\exp\left(-\eta\frac{M_\epsilon}{2} - \eta\sqrt{\frac{2}{H}}\right) - 2b\gamma\exp\left(-\eta\frac{M_\epsilon}{2}\right) \\
> \;& 0.
\end{aligned}$$

Case 3: $\frac{M_\epsilon}{2} < r \le M_\epsilon$. Since in this case $g(r) = 0$, by (42) and (45), it is easy to see that $\Delta w^- + f(w^-) \ge 0$.

Therefore it is routine to check that $w^-(z)$ is a weak lower solution to (40). On the other hand side, $w^+(z) \equiv \alpha^+$ is an upper solution to (40) and obviously $w^+(z) > w^-(z)$ in $B(0;\ M_\epsilon)$. Then due to standard results (cf. e.g., [18, Theorem 3.6] and [11, Theorem 4]), there exists a solution, denoted by w_2, to (40) and $w^-(z) \le w_2(z) \le w^+(z)$ in $B(0;\ M_\epsilon)$. □

Now it is time to put together Lemmas 3.2 and 3.3 and finish the last piece of the proof of Proposition 1. Recalling (39) and (40), from Lemma 3.3, we automatically get a solution, denoted by v_2, to (39) by setting

$$v_2(x) = w_2\left(\frac{h_{\min}}{\epsilon}(x - x_1)\right).$$

Moreover, because of Lemma 3.2, we may derive the following estimate

$$u_\epsilon(x) \ge v_2(x) \quad \text{for } x \in B(x_1;\ \frac{2}{3}d(x_0) - \epsilon A),$$

by applying the similar arguments as we have used to prove Lemma 3.2. Notice that $x_1 \in \partial B(x_0;\ \frac{1}{3}d(x_0))$ and $M_\epsilon = h_{\min}\left(\frac{2}{3}\frac{d(x_0)}{\epsilon} - A\right)$. Then it follows that

$$\begin{aligned}
& u_\epsilon(x_0) \ge v_2(x_0) = w_2\left(\frac{h_{\min}}{\epsilon}(x_0 - x_1)\right) \\
\ge \;& w^-\left(\frac{h_{\min}}{\epsilon}|x_0 - x_1|\right) = \alpha^+ - b\exp(\eta h_{\min}A)\exp\left(-\frac{\eta h_{\min}}{3}\frac{d(x_0)}{\epsilon}\right),
\end{aligned}$$

where w^- is the lower solution defined in (44). The inequality $u_\epsilon < \alpha^+$ is obtained as follows. Suppose that the set $\{x \in \Omega \mid u_\epsilon > \alpha^+\}$ is not empty. A contradiction may be derived by applying maximum principle in one connected component of this set. Moreover, the possibility that $u_\epsilon(x) = \alpha^+$ at some point in Ω contradicts to strong maximum principle. Now the inequality (28) is established when we assume that $u_\epsilon > 0$ in $B(x_0;\ d(x_0))$. Obviously, the inequality (29) can be proved similarly when assuming $u_\epsilon < 0$ in $B(x_0;\ d(x_0))$. The proof of Proposition 1 is complete.

4. **Characterize the location of transition layers.** In this section, we will prove a stronger result, that is Theorems 4.1 and 4.2, which includes Theorem 1.3.

Theorem 4.1. *Let S be a C^3 hypersurface in Ω, and Λ be a connected open set in Ω such that $S \cap \Lambda$ is nonempty. If there exist $\eta > 0$, $\{\epsilon_n > 0\}_{n=1}^\infty$, and $\{\delta_n > 0\}_{n=1}^\infty$ such that*

(1) $\lim\limits_{n\to\infty} \epsilon_n = 0$ *and* $\lim\limits_{n\to\infty} \delta_n = 0$.
(2) For every $\epsilon = \epsilon_n$, (2) has a solution u_{ϵ_n} with the property that at least one connected component of $\Gamma_n \cap \Lambda$ is contained in $S_\eta \cap \Lambda$.
(3) $\Gamma_n \cap (S_\eta \cap \Lambda) \subset S_{\delta_n} \cap \Lambda$.

Then
$$\frac{\partial h}{\partial \mu} - \kappa(N-1)h = 0 \ \text{ on } \ S \cap \Lambda,$$
where $\Gamma_n \equiv \{x \in \Omega \mid u_{\epsilon_n}(x) = 0\}$, $S_\eta \equiv \left\{x \in R^N \mid dist(x,\ S) < \eta\right\}$ and μ is the unit normal vector to $S \cap \Lambda$ and κ is the mean curvature of $S \cap \Lambda$.

Remark 6. Theorem 4.1 reveals that the equation (2) only allows layers near a curve satisfying the hypothesis (H1). Here the term "layer" includes clustering layers such as in [11]. Note that the hypersurface S in Theorem 4.1 can intersect the boundary of Ω.

Theorem 4.2. *Suppose that $S \cap \Lambda$ mentioned in Theorem 4.1 is a closed hypersurface and the conditions (1), (2), (3) in Theorem 4.1 still hold. Denote $\Upsilon \equiv S \cap \Lambda$ and choose η small enough such that $\Upsilon_\eta \equiv \left\{x \in R^N \mid dist(x,\ \Upsilon) < \eta\right\} \subset \Lambda$. If*
$$\frac{\partial^2 h}{\partial \mu^2} \neq (N-1)^2\kappa^2 h + h\sum_{i=1}^{N-1} \lambda_i^2 \tag{46}$$
for every point on Υ, then we have $\Gamma_n \cap \Upsilon_\eta \subset \Upsilon_{C\epsilon_n|\log \epsilon_n|^2}$, when n is sufficiently large, where the constant $C > 0$ is independent of n, μ is the unit inner normal vector to Υ, κ is the mean curvature of Υ and $\lambda_1, \lambda_2, ..., \lambda_{N-1}$ are the principal curvatures of S.

The rest of the section is devoted to the proofs of Theorems 4.1 and 4.2. Since the proof is a long story, we divide it into three subsections.

4.1. **Preliminaries.** Suppose that the statement in Theorem 4.1 is not true. Then there exists $P \in S \cap \Lambda$ such that either $\frac{\partial h}{\partial \mu} - \kappa(N-1)h < 0$ or $\frac{\partial h}{\partial \mu} - \kappa(N-1)h > 0$ holds at P. W.l.o.g., we assume that
$$\frac{\partial h}{\partial \mu} - \kappa(N-1)h > 0 \quad \text{at } P. \tag{47}$$
A contradiction will be derived later in Section 4.2.

W.l.o.g., suppose that $S \cap \Lambda$ divide the domain Λ into two subdomains. Let D denote the one which makes the unit normal vector μ at P to be the unit inner normal vector of ∂D, and

$$S(\delta_n) \equiv \{x \in D \mid dist(x, S) = \delta_n\}\,.$$

We may choose n larger to guarantee that δ_n is small enough when necessary. There exist a C^3 hypersurface $\widetilde{S}$ in D and a small constant $\delta > 0$ such that the following properties hold:

(P1) $\widetilde{S}$ touches $S(\delta_n)$ only at the point P^* which is closest to P. See Fig. 4.1.

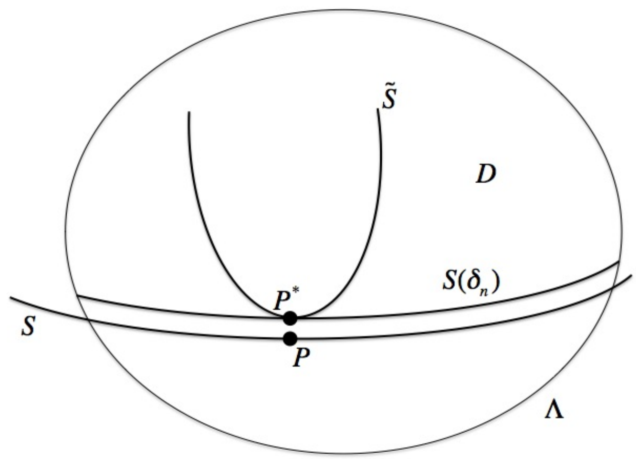

Fig. 4.1.

(P2) There exists a C^3 differentiable homeomorphism

$$\gamma : U \;\to\; V \cap \widetilde{S},$$

where U is open in R^{N-1} and V is open in Λ. Moreover, $P^* \in \gamma(U)$ and

$$\sigma(s,t) \equiv \gamma(s) + t\mu[\gamma(s)]$$

is a C^2 differentiable homeomorphism from $U \times (-2\delta, 2\delta)$ to its image.

(P3) Due to (47), there exists a small positive constant m_0 such that

$$\frac{1}{\widetilde{h}^2}\left(\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h}\right) > m_0 \quad \text{on } U \times (-2\delta, 2\delta), \tag{48}$$

where

$$\widetilde{h}(s,t) = h(\sigma(s,t)) = h(x_1, ..., x_N).$$

Here we may need make U and V smaller if necessary.

(P4) There exists an open set $Q \subset U$ such that $P^* \in \gamma(Q)$ and

$$dist(x, S(\delta_n)) > 2\delta \quad \text{for } x \in \{\sigma(s,t) \mid s \in \partial Q,\ t = 0\}\,.$$

(P5) $\sigma(Q \times [-\delta, \delta]) \subset S_{\eta/2} \cap \Lambda$.

According to the assumptions of Theorem 4.1, either $u_{\epsilon_n}(x) < 0$ or $u_{\epsilon_n}(x) > 0$ in $(S_\eta \setminus S_{\delta_n}) \cap D$. W.l.o.g. we assume that $u_{\epsilon_n}(x) < 0$ in $(S_\eta \setminus S_{\delta_n}) \cap D$. In this case, Proposition 1 shows that there exist $\eta_1 > 0$, $C_1 > 0$ such that

$$0 \le u_{\epsilon_n}(x) - \alpha^- < C_1 \exp\left(-\frac{\eta_1 d(x)}{\epsilon_n}\right), \tag{49}$$

for ϵ_n sufficiently small. From now on, we set $\epsilon = \epsilon_n$ for simplicity and always assume that ϵ is sufficiently small.

For each $t_0 \in [-2\delta_n, \frac{\delta}{2}]$, define

$$u^{(t_0)}(x) \equiv \widetilde{u}^{(t_0)}(s,t) \equiv V^+\left(-\int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau\right), \tag{50}$$

where $s \in Q$, $t \in [-\delta, \delta]$, V^+ is defined in (15) and $x = \sigma(s,t) \in \sigma(Q \times [-\delta,\delta])$.

Notice that, by (12), (15) and Lemmas 2.3, 2.4,

$$u^{(t_0)}(x) \ge v^+(-\tau_1) \ge \phi(-\tau_1) \ge \alpha^- + A_1 \epsilon^{\frac{5\hat{R}}{3R}}, \tag{51}$$

for $x \in \sigma(Q \times [-\delta,\delta])$.

Denote

$$\begin{aligned}
\Sigma &= \sigma(Q \times [-\delta,\delta]),\\
\Sigma_0 &= \left\{ x = \sigma(s,t) \in \Sigma \;\middle|\; -\epsilon\tau_2 < \int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau < \epsilon\tau_1 \right\},\\
\Sigma_1 &= \left\{ x = \sigma(s,t) \in \Sigma \;\middle|\; \int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau > \epsilon\tau_1 \right\},\\
\Sigma_2 &= \left\{ x = \sigma(s,t) \in \Sigma \;\middle|\; \int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau < -\epsilon\tau_2 \right\}.
\end{aligned}$$

We includes Fig. 4.2 for the convenience of readers.

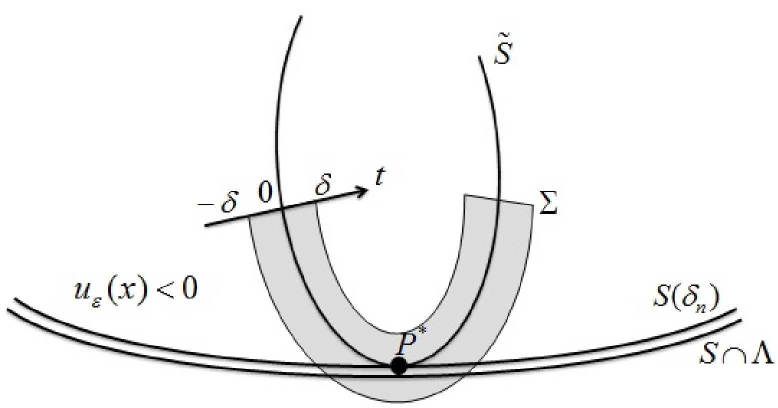

Fig. 4.2.

Remark 7. We have carefully prepared the environment where our arguments will take place. Particularly, in (50), we construct a series of functions $u^{(t_0)}(x)$ which are only defined in Σ. Locally, the functions $u^{(t_0)}(x)$ behave like upper solutions of (2). However, when we try to compare the functions $u^{(t_0)}(x)$ with the solution of

(2) to derive a contradiction, we have to worry about the boundary of Σ. This is why Σ is chosen in this special way.

4.2. **Proof of Theorem 4.1.** To avoid the distraction of complicated calculations and make our idea transparent, we present our main steps in the form of lemmas and derive a contradiction to finish the proof of Theorem 4.1 first. Then we provide proofs of these lemmas.

Lemma 4.3. *For any $t_0 \in [-2\delta_n, \frac{\delta}{2}]$, it holds that*

$$\epsilon^2 \Delta u^{(t_0)}(x) + h^2(x) f(u^{(t_0)}(x)) \leq 0 \quad \textit{in } \Sigma_0 \cup \Sigma_1 \cup \Sigma_2,$$

for sufficiently small $\epsilon > 0$.

Remark 8. By the definition of $u^{(t_0)}$, $t_0 \in [-2\delta_n, \frac{\delta}{2}]$, it is routine to check that

$$\frac{\partial u^{(t_0)}(x)}{\partial x_i} = 0, \quad \text{for } x \in \left\{ x = \sigma(s,t) \in \Sigma \,\middle|\, \int_{t_0}^{t} \widetilde{h}(s,\tau) d\tau = \epsilon \tau_1 \right\},$$

where $x = (x_1, ..., x_N)$, $i = 1, ..., N$.

Lemma 4.4. *Let $t_0 \in [-2\delta_n, \frac{\delta}{2}]$. For $\epsilon > 0$ sufficiently small, we have*

$$u^{(t_0)}(x) > u_\epsilon(x), \quad x \in \partial\Sigma = \Pi_1 \cup \Pi_2 \cup \Pi_3, \tag{52}$$

where

$$\begin{aligned} \Pi_1 &= \left\{ x = \sigma(s,t) \,\middle|\, s \in Q, t = -\delta \right\}, \\ \Pi_2 &= \left\{ x = \sigma(s,t) \,\middle|\, s \in Q, t = \delta \right\}, \\ \Pi_3 &= \left\{ x = \sigma(s,t) \,\middle|\, s \in \partial Q, -\delta \leq t \leq \delta \right\}. \end{aligned}$$

Lemma 4.5. *For $\epsilon > 0$ sufficiently small,*

$$u^{(\frac{\delta}{2})}(x) > u_\epsilon(x), \quad x \in \Sigma.$$

Lemma 4.6. *There exists $\widetilde{x} \in \sigma(Q \times (-\delta, \delta))$, such that $u^{(-2\delta_n)}(\widetilde{x}) < u_\epsilon(\widetilde{x})$.*

First of all, Lemma 4.4 guarantees that at the boundary of Σ, $u^{(t_0)}(x) > u_\epsilon(x)$ for $t_0 \in [-2\delta_n, \frac{\delta}{2}]$. Then, setting $t_0 = \frac{\delta}{2}$, we have $u^{(\frac{\delta}{2})}(x) > u_\epsilon(x)$ for $x \in \Sigma$, because of Lemma 4.5. Therefore, when we let t_0 decreases from $\frac{\delta}{2}$ to $-2\delta_n$, thanks to the strong maximum principle and Hopf boundary lemma, Lemma 4.3 and Remark 8 following it imply that for any $t_0 \in [-2\delta_n, \frac{\delta}{2}]$,

$$u^{(t_0)}(x) > u_\epsilon(x), \quad x \in \Sigma,$$

which obviously contradicts Lemma 4.6. Theorem 4.1 is proved.

Now we only need prove these lemmas. The proof of Lemma 4.3 is similar to that of Lemma 2.6. Proposition 1 is used frequently in the proofs of Lemmas 4.4 and 4.5.

Proof of Lemma 4.3. According to the definition of $u^{(t_0)}(x)$ in (50), it is easy to check that

$$\epsilon^2 \Delta u^{(t_0)}(x) + h^2(x) f(u^{(t_0)}(x)) \leq 0 \quad \text{in } \Sigma_1 \cup \Sigma_2.$$

Then we only need show that

$$\epsilon^2 \Delta u^{(t_0)}(x) + h^2(x) f(u^{(t_0)}(x)) \leq 0 \quad \text{in } \Sigma_0. \tag{53}$$

The proof is similar to that of Lemma 2.6. Therefore, we mainly explain how to modify the proof of Lemma 2.6.

By setting

$$y = -\int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau, \quad z = \frac{y}{\epsilon}$$

and repeating similar calculation we have carried on to obtain (21), we get

$$\begin{aligned} & \frac{1}{h^2}\left[\epsilon^2 \Delta u^{(t_0)}(x) + h^2 f(u^{(t_0)}(x))\right] \\ \leq \quad & \epsilon m_1(s,t)\left[\phi'(z) + \frac{R}{2}\beta\epsilon^2\theta'(\frac{R}{2}z)\right] + \beta\epsilon^2\left[\frac{R^2}{4}\theta''(\frac{R}{2}z) + f'(\xi(z))\theta(\frac{R}{2}z)\right] \\ & + C_3\beta\epsilon^4|\log\epsilon|^2\left(\left|\theta''(\frac{R}{2}z)\right| + \left|\theta'(\frac{R}{2}z)\right|\right) \end{aligned} \tag{54}$$

for some positive constant C_3, where

$$\phi(z) \leq \xi(z) \leq \phi(z) + \beta\epsilon^2\theta(\frac{R}{2}z) \leq \alpha^+,$$

and

$$\begin{aligned} m_1(s,t) \quad \equiv \quad & \frac{1}{\widetilde{h}^2}\Bigg\{ -\widetilde{h}_t - \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h} \\ & + \frac{1}{\epsilon}\sum_{j,k=1}^{N-1} g^{jk}\frac{\phi_{zz}}{\phi_z}\int_{t_0}^{t}\widetilde{h}_{s_j}d\tau\int_{t_0}^{t}\widetilde{h}_{s_k}d\tau \\ & - \frac{1}{\sqrt{g}}\sum_{j,k=1}^{N-1}\left[\partial_j(g^{jk}\sqrt{g})\int_{t_0}^{t}\widetilde{h}_{s_k}d\tau + g^{jk}\sqrt{g}\int_{t_0}^{t}\widetilde{h}_{s_j s_k}d\tau\right]\Bigg\}. \end{aligned}$$

Recall that $x = \sigma(s,t) \in \Sigma_0$ means that $-\tau_1 \leq z = -\frac{1}{\epsilon}\int_{t_0}^{t}\widetilde{h}(s,\tau)d\tau \leq \tau_2$. Combined with Lemma 2.4, we have

$$|t - t_0| \leq C_3'\epsilon|\log\epsilon|.$$

Then it follows from (48) that

$$m_1(s,t) \leq -m_0 + \frac{C_3''}{\epsilon}|t-t_0|^2 + C_3''|t-t_0| \leq -\frac{m_0}{2} \tag{55}$$

for $-\tau_1 \leq z \leq \tau_2$.

We only prove (53) for $0 \leq z \leq \tau_2$, since the case $-\tau_1 \leq z \leq 0$ can be verified in the same way. Now let us come back to (54) and handle the following two cases separately.

Case 1: $0 \leq z \leq N^*$, where N^* is defined in (17). It follows that

$$\left|\frac{R^2}{4}\theta''(\frac{R}{2}z) + f'(\xi(z))\theta(\frac{R}{2}z)\right| \leq C_4 \exp\left(\frac{R}{2}N^*\right). \tag{56}$$

Hence Lemma 2.3, (54), (55) and (56) together imply that

$$\begin{aligned} & \frac{1}{h^2}\left[\epsilon^2\Delta U_L^+ + h^2 f(U_L^+)\right] \\ \leq \quad & -\frac{A_1 m_0\epsilon}{2}\exp(-\widehat{R}N^*) + \beta C_4\epsilon^2\exp\left(\frac{R}{2}N^*\right) + O(\epsilon^3) < 0 \end{aligned}$$

for $\epsilon > 0$ sufficiently small.

Case 2: $N^* \leq z \leq \tau_2$. By the same way as we obtain (27), we have

$$\frac{R^2}{4}\theta''(\frac{R}{2}z) + f'(\xi(z))\theta(\frac{R}{2}z) < -\frac{R^2}{4}\theta(\frac{R}{2}z).$$

This inequality, (54), (55) and Lemma 2.4 together yield that

$$\begin{aligned}\frac{1}{h^2}\left[\epsilon^2\Delta U_L^+ + h^2 f(U_L^+)\right] &\leq -\frac{R^2}{4}\beta\epsilon^2\theta(\frac{R}{2}z) + C\epsilon^4|\log\epsilon|^2\epsilon^{-\frac{5}{6}} \\ &\leq -\frac{R^2}{4}\beta\epsilon^2\theta(\frac{R}{2}z) + O(\epsilon^3) < 0,\end{aligned}$$

for $\epsilon > 0$ sufficiently small.

The proof is complete. □

The proof of Lemma 4.4 in the following might look confusing. Intuitively, on the boundary Π_1, $u^{(t_0)}(x) = \alpha^+$ according to its definition, hence $u^{(t_0)}(x) > u_\epsilon(x)$. While on the boundary $\Pi_2 \cup \Pi_3$, u_ϵ is negative and all the points are away from zero level set of the solution u_ϵ, thus u_ϵ is very close to α^- due to Proposition 1 and thus it is smaller than $u^{(t_0)}(x)$ on $\Pi_2 \cup \Pi_3$. See Fig 4.3 as follows.

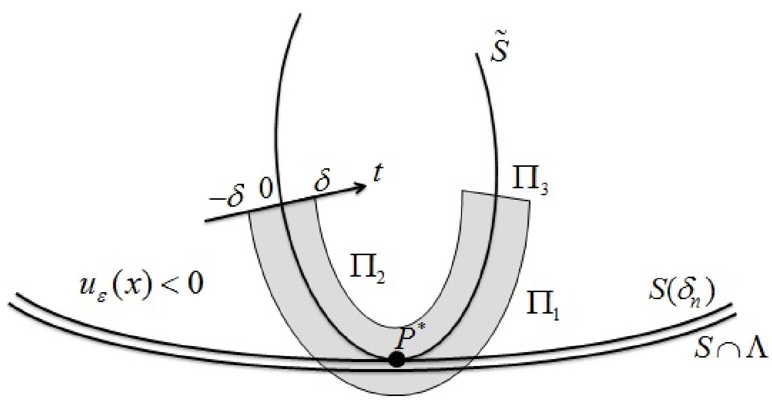

Fig. 4.3.

Proof of Lemma 4.4. First recall that

$$u^{(t_0)}(x) \equiv \widetilde{u}^{(t_0)}(s,t) \equiv V^+\left(-\int_{t_0}^{t}\widetilde{h}(s,\tau)d\tau\right)$$

for $s \in Q,\ t \in [-\delta,\delta],\ x = \sigma(s,t)$.

Observe that when $x \in \Pi_1$, $-\int_{t_0}^{-\delta}\widetilde{h}(s,\tau)d\tau \geq (\delta - 2\delta_n)h_{\min} > \epsilon\tau_2$ by Lemma 2.4. This, according to the definition of V^+ in (15), implies that $u^{(t_0)}(x) = \alpha^+$, for $x \in \Pi_1$. While $u_\epsilon < \alpha^+$ by Proposition 1. Thus $u^{(t_0)}(x) > u_\epsilon(x)$ for $x \in \Pi_1$.

Next, consider the case that $x \in \Pi_2$. On the one hand side, $u^{(t_0)}(x) \geq \alpha^- + A_1\epsilon^{\frac{5\widehat{R}}{3R}}$ due to (51). On the other hand side, noticing that $d(x) \geq dist(x, S(\delta_n)) \geq dist(x, \widetilde{S}) = \delta$, by (49), we have

$$u_\epsilon(x) < \alpha^- + C_1\exp\left(-\frac{\eta_1\delta}{\epsilon}\right).$$

Therefore, $u^{(t_0)}(x) > u_\epsilon(x)$ for $x \in \Pi_2$.

Now it remains to show that $u^{(t_0)}(x) > u_\epsilon(x)$ for $x \in \Pi_3$. For any fixed $x = \sigma(q,t) \in \Pi_3$, consider $dist(x, S(\delta_n)) = dist(\sigma(q,t), S(\delta_n))$. Let $P_n \in S(\delta_n)$ denote the point that satisfies $dist(\sigma(q,t), S(\delta_n)) = |P_n - \sigma(q,t)|$. By the properties (P4) and (P5) from the beginning of this subsection, we have

$$|P_n - \sigma(q,0)| \geq dist(\sigma(q,0), S(\delta_n)) > 2\delta \text{ and } d(x) \geq dist(x, S(\delta_n)).$$

Hence it follows that

$$d(x) \geq dist(x, S(\delta_n)) = |P_n - \sigma(q,t)| \geq |P_n - \sigma(q,0)| - |\sigma(q,t) - \sigma(q,0)| > \delta,$$

which, again combined with (49), yields that

$$u_\epsilon(x) < \alpha^- + C_1 \exp\left(-\frac{\eta_1 \delta}{\epsilon}\right).$$

Then because of (51), it is easy to see that $u^{(t_0)}(x) > u_\epsilon(x)$ for $x \in \Pi_3$. The proof is complete. □

Proof of Lemma 4.5. For convenience, let us write down the definition of $u^{(\frac{\delta}{2})}(x)$ as follows

$$u^{(\frac{\delta}{2})}(x) \equiv \widetilde{u}^{(\frac{\delta}{2})}(s,t) \equiv V^+\left(-\int_{\frac{\delta}{2}}^{t} \widetilde{h}(s,\tau)d\tau\right)$$

for $s \in Q,\ t \in [-\delta, \delta],\ x = \sigma(s,t)$. When $x = \sigma(s,t) \in \sigma\left(Q \times [-\delta, \frac{\delta}{4}]\right)$, it is easy to see that, by Lemma 2.4,

$$-\int_{\frac{\delta}{2}}^{t} \widetilde{h}(s,\tau)d\tau \geq \frac{\delta}{4} h_{\min} > \epsilon\tau_2.$$

Thus, from the definition of V^+ in (15), we have

$$u^{(\frac{\delta}{2})}(x) = \alpha^+, \quad \text{for } x = \sigma(s,t) \in \sigma\left(Q \times [-\delta, \frac{\delta}{4}]\right).$$

On the other hand side, $u_\epsilon < \alpha^+$ by Proposition 1. Therefore,

$$u^{(\frac{\delta}{2})}(x) > u_\epsilon(x), \quad \text{for } x \in \sigma\left(Q \times [-\delta, \frac{\delta}{4}]\right).$$

For $x = \sigma(s,t) \in \sigma\left(Q \times (\frac{\delta}{4}, \delta]\right)$, noticing that $d(x) \geq dist(x, S(\delta_n)) \geq \frac{\delta}{4}$, we get, by using (49),

$$u_\epsilon(x) < \alpha^- + C_1 \exp\left(-\frac{\eta_1 \delta}{4\epsilon}\right).$$

Then, because of (51), it is obvious that

$$u^{(\frac{\delta}{2})}(x) > u_\epsilon(x), \quad \text{for } x \in \sigma\left(Q \times (\frac{\delta}{4}, \delta]\right).$$

□

Proof of Lemma 4.6. By the definition of $u^{(-2\delta_n)}(x)$, it is routine to check that

$$u^{(-2\delta_n)}(x) < 0 \quad \text{for } x \in \sigma(Q \times (-2\delta_n, 0)).$$

However, u_ϵ has zeros in $\sigma(Q \times (-2\delta_n, 0))$. Hence there exists $\widetilde{x} \in \sigma(Q \times (-2\delta_n, 0))$, such that $u^{(-2\delta_n)}(\widetilde{x}) < u_\epsilon(\widetilde{x})$. □

4.3. **Proof of Theorem 4.2.** For the rest of this section, we focus on proving Theorem 4.2. The idea in the proof of Theorem 4.2 is quite similar to that in the proof of Theorem 4.1. We will also assume that our claim is false, construct a series of proper functions based on this assumption, and then derive a contradiction by comparing these functions with the solution of the problem (2). The best part of the proof is the construction of proper functions. To avoid unnecessary repetition and keep the proof concise, we only emphasize the different parts and skip some calculations when they are similar to previous proofs.

Now let us begin our proof. Suppose that the statement of Theorem 4.2 is not true. This means that there exists $\{M_n > 0\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} M_n = +\infty,\ M_n < \frac{\delta_n}{\epsilon_n|\log\epsilon_n|^2},$$

and

$$\Gamma_n \cap \left(\Upsilon_{\delta_n} \setminus \Upsilon_{M_n\epsilon_n|\log\epsilon_n|^2}\right) \neq \emptyset,$$

for n large. W.l.o.g., assume that

$$\Gamma_n \cap \left(\Upsilon_{\delta_n} \setminus \Upsilon_{M_n\epsilon_n|\log\epsilon_n|^2}\right) \cap D \neq \emptyset, \tag{57}$$

where D denotes a domain bounded by Υ.

Since $\Upsilon = S \cap \Lambda$ is a C^3 closed hypersurface, there exists the local coordinate system in Υ_η introduced in subsection 2.1. Choose η smaller if necessary. By Theorem 4.1, we already have

$$\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h} = 0 \quad \text{at } t = 0. \tag{58}$$

Moreover, the assumption (46) is equivalent to

$$\frac{\partial}{\partial t}\left(\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h}\right) \neq 0 \quad \text{at } t = 0.$$

W.l.o.g., we assume that

$$\frac{\partial}{\partial t}\left(\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h}\right) > 0 \quad \text{at } t = 0. \tag{59}$$

It follows from (58) and (59) that, for any large constant $A > 0$ to be determined later, there exists n sufficiently large such that $M_n >> 0$ and

$$\widetilde{h}_t + \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h} > A\epsilon_n|\log\epsilon_n|^2 \quad \text{for } t > \frac{M_n}{2}\epsilon_n|\log\epsilon_n|^2. \tag{60}$$

By the assumptions of Theorem 4.2, either $u_{\epsilon_n}(x) < 0$ or $u_{\epsilon_n}(x) > 0$ on $(\Upsilon_\eta \setminus \Upsilon_{\delta_n})\cap D$, where D denotes a domain bounded by Υ. W.l.o.g. we assume that $u_{\epsilon_n}(x) < 0$.

Set $\epsilon = \epsilon_n$ for simplicity. For each $t_0 \in [M_n\epsilon|\log\epsilon|^2, \eta/4]$, define

$$u_1^{(t_0)}(x) \equiv \widetilde{u}_1^{(t_0)}(s,t) \equiv V^+\left(-\int_{t_0}^{t}\widetilde{h}(s,\tau)d\tau\right),$$

where $x = \sigma(s,t) \in \overline{\Upsilon_{\eta/2}}$, and V^+ is defined in (15). We also have the estimate (51) for $u_1^{(t_0)}(x)$, that is,

$$u_1^{(t_0)}(x) \geq \alpha^- + A_1\epsilon^{\frac{5\widehat{R}}{3R}} \quad \text{for } x \in \overline{\Upsilon_{\eta/2}}.$$

Note that the estimate (49)

$$0 \le u_\epsilon(x) - \alpha^- < C_1 \exp\left(-\frac{\eta_1 d(x)}{\epsilon}\right)$$

for the solution $u_\epsilon(x)$ of (2) still holds.

First, denote

$$\begin{aligned}
\Sigma_0' &= \left\{ x = \sigma(s,t) \in \Upsilon_{\eta/2} \;\middle|\; -\epsilon\tau_2 < \int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau < \epsilon\tau_1 \right\}, \\
\Sigma_1' &= \left\{ x = \sigma(s,t) \in \Upsilon_{\eta/2} \;\middle|\; \int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau > \epsilon\tau_1 \right\}, \\
\Sigma_2' &= \left\{ x = \sigma(s,t) \in \Upsilon_{\eta/2} \;\middle|\; \int_{t_0}^{t} \widetilde{h}(s,\tau)d\tau < -\epsilon\tau_2 \right\}.
\end{aligned}$$

Then similar to Lemma 4.3, we claim that *for any* $t_0 \in [M_n\epsilon|\log\epsilon|^2, \eta/4]$

$$\epsilon^2 \Delta u_1^{(t_0)}(x) + h^2(x) f(u_1^{(t_0)}(x)) \le 0 \quad \text{in } \Sigma_0' \cup \Sigma_1' \cup \Sigma_2', \tag{61}$$

provided $\epsilon > 0$ *is sufficiently small.*

Obviously, in $\Sigma_1' \cup \Sigma_2'$,

$$\epsilon^2 \Delta u_1^{(t_0)}(x) + h^2(x) f(u_1^{(t_0)}(x)) \le 0.$$

For $x \in \Sigma_0'$, by the same arguments and calculations in Lemma 4.3, we get

$$\begin{aligned}
&\frac{1}{h^2}\left[\epsilon^2 \Delta u_1^{(t_0)}(x) + h^2 f(u_1^{(t_0)}(x))\right] \\
\le\;& \epsilon m_2(s,t)\left[\phi'(z) + \frac{R}{2}\beta\epsilon^2\theta'(\frac{R}{2}z)\right] + \beta\epsilon^2\left[\frac{R^2}{4}\theta''(\frac{R}{2}z) + f'(\xi(z))\theta(\frac{R}{2}z)\right] \\
&+ C_5\beta\epsilon^4|\log\epsilon|^2\left(\left|\theta''(\frac{R}{2}z)\right| + \left|\theta'(\frac{R}{2}z)\right|\right),
\end{aligned}$$

for some positive constant C_5, where

$$\phi(z) \le \xi(z) \le \phi(z) + \beta\epsilon^2\theta(\frac{R}{2}z) \le \alpha^+,$$

and

$$\begin{aligned}
m_2(s,t) \equiv\;& \frac{1}{\widetilde{h}^2}\Bigg\{ -\widetilde{h}_t - \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial t}\widetilde{h} \\
&+ \frac{1}{\epsilon}\sum_{j,k=1}^{N-1} g^{jk}\frac{\phi_{zz}}{\phi_z}\int_{t_0}^{t}\widetilde{h}_{s_j}d\tau \int_{t_0}^{t}\widetilde{h}_{s_k}d\tau \\
&- \frac{1}{\sqrt{g}}\sum_{j,k=1}^{N-1}\left[\partial_j(g^{jk}\sqrt{g})\int_{t_0}^{t}\widetilde{h}_{s_k}d\tau + g^{jk}\sqrt{g}\int_{t_0}^{t}\widetilde{h}_{s_j s_k}d\tau\right]\Bigg\}.
\end{aligned}$$

Moreover, $|t - t_0| \le C_5'\epsilon|\log\epsilon|$. Then, combined with Lemma 2.4 and the assumption (60), we obtain that

$$m_2(s,t) \le -A\epsilon|\log\epsilon|^2 + \frac{C_5''}{\epsilon}|t-t_0|^2 + C_5''|t-t_0| \le -\frac{A}{2}\epsilon|\log\epsilon|^2$$

for $-\tau_1 \leq z \leq \tau_2$, if we choose $A > 0$ large enough such that $A > 3(C_5')^2 C_5''$. Using the above estimate, the claim can be proved by the same argument as those used in proving Lemma 4.3. We omit the details.

We also remark that it follows from the definition of $u_1^{(t_0)}$, $t_0 \in [M_n \epsilon |\log \epsilon|^2, \eta/4]$ that

$$\frac{\partial u_1^{(t_0)}(x)}{\partial x_i} = 0, \quad \text{for } x \in \left\{ \sigma(s,t) \in \Upsilon_{\eta/2} \;\middle|\; \int_{t_0}^{t} \widetilde{h}(s,\tau) d\tau = \epsilon \tau_1 \right\}, \tag{62}$$

where $x = (x_1, ..., x_N)$, $i = 1, ..., N$.

Next, notice that since Υ is closed, the boundary of $\Upsilon_{\eta/2}$ is

$$\Pi \equiv \left\{ x = \sigma(s,t) \in \overline{\Upsilon_{\eta/2}} \mid t = -\frac{\eta}{2} \ \text{ or } t = \frac{\eta}{2} \right\}.$$

By following similar arguments in Lemma 4.4, we derive that

$$u_1^{(t_0)}(x) > u_\epsilon(x), \quad x \in \Pi. \tag{63}$$

Moreover, similar to Lemma 4.5, we claim that *for ϵ sufficiently small,*

$$u_1^{(\eta/4)}(x) > u_\epsilon(x) \quad \textit{for } \overline{\Upsilon_{\eta/2}}. \tag{64}$$

Recall that $\Gamma_n \cap \Upsilon_\eta \subset \Upsilon_{\delta_n}$ from the assumptions of Theorem 4.2. Then the claim can be verified by applying similar arguments in Lemma 4.5. We omit the details.

Finally, from the assumption (57), it is easy to show that there exists $\widetilde{x}_1 \in \Upsilon_{\eta/2}$, such that

$$u_1^{(M_n \epsilon |\log \epsilon|^2)}(\widetilde{x}_1) < u_\epsilon(\widetilde{x}_1). \tag{65}$$

This is a similar result to Lemma 4.6.

Therefore, due to the strong maximum principle and Hopf boundary lemma, (61), (62), (63) and (64) together imply that for any $t_0 \in [M_n \epsilon |\log \epsilon|^2, \eta/4]$, we have

$$u_1^{(t_0)}(x) > u_\epsilon(x) \quad \text{for } \overline{\Upsilon_{\eta/2}}.$$

This is a contradiction to (65). The proof of Theorem 4.2 is complete.

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