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EFFECTIVE BOUNDARY CONDITIONS OF THE HEAT EQUATION ON A BODY COATED BY FUNCTIONALLY GRADED MATERIAL

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Abstract. We consider the linear heat equation on a bounded domain, which has two components with a thin coating surrounding a body (of metallic nature), subject to the Dirichlet boundary condition. The coating is composed of two layers, the pure ceramic part and the mixed part. The mixed part is considered to be functionally graded material (FGM) that is meant to make a smooth transition from being metallic to being ceramic. The diffusion tensor is isotropic on the body, and allowed to be anisotropic on the coating; and the size of diffusion tensor may differ significantly in these components. We find effective boundary conditions (EBCs) that are approximately satisfied by the solution of the heat equation on the boundary of the body. A concrete example is considered to study the effect of FGM coating. We also provide numerical simulations to verify our theoretical results.

1. **Introduction.** In [8], the authors studied the asymptotic behavior of the linear heat equation on a bounded domain Ω, which is composed of an isotropically conducting body Ω_1 (say, of metallic nature) surrounded by a thin coating Ω_2 (say, of ceramic nature) that is allowed to be anisotropic. See Figure 1. The physical model includes space crafts and turbine engine blades protected by thermal insulators. These (ceramic) coatings protect the space crafts or turbine engine blades from high temperature experienced during operation. Effective boundary conditions (EBCs) are obtained on $\partial\Omega_1$, as the thickness of the coating shrinks. EBCs are approximately satisfied by the solution to the heat equation on the boundary of the body Ω_1. It turns out that the EBC is completely determined by the scaling relationship between the thermal tensor and the thickness of the coating.

Finding EBC enables us to see the effect of the coating. For example, homogeneous Neumann EBC is desirable for perfect insulation of the body Ω_1. Furthermore, it is computationally challenging to solve the heat equation on Ω, due to smallness of the thermal conductivity on Ω_2 and the thickness of the coating. However, if we know the EBC on $\partial\Omega_1$, we can simply solve the equation on Ω_1 with the EBC, which involves no small scales.

The mathematical model studied is the following. Let $\Omega_1 \subset \mathbb{R}^n$ be a bounded domain with C^2-smooth boundary. The coating layer Ω_2 is uniformly thick with

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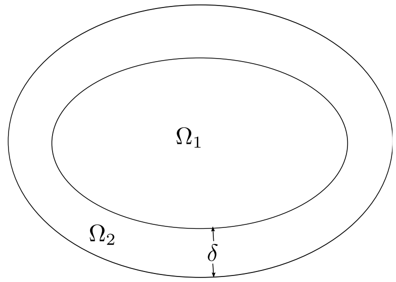


FIGURE 1. $\Omega = \overline{\Omega}_1 \cup \Omega_2$. The coating Ω_2 is uniformly thick with thickness δ.

thickness δ, i.e., $\Omega_2 = \{x \notin \Omega_1 \,|\, 0 < \text{dist}(x, \partial\Omega_1) < \delta\}$. Let $\Omega = \overline{\Omega}_1 \cup \Omega_2$. The thermal tensor of Ω is given by

$$A(x) = (a_{ij}(x))_{n\times n} = \begin{cases} kI_{n\times n}, & x \in \Omega_1, \\ \sigma\left(\overline{a}_{ij}(x)\right)_{n\times n}, & x \in \Omega_2, \end{cases} \tag{1.1}$$

where both k and σ are positive constants, $I_{n\times n}$ is the identity matrix and $(\overline{a}_{ij})$ is a symmetric positive-definite matrix. σ is a parameter that measures the thermal conductivity in all directions; if it is small, it means that the coating is a good thermal barrier.

The following initial boundary value problem is investigated in [8].

$$\begin{cases} u_t = \nabla\cdot(A(x)\nabla u) + f(x,t), & (x,t) \in Q_T, \\ u = 0, & (x,t) \in S_T, \\ u = \varphi(x), & x \in \Omega,\ t = 0, \end{cases} \tag{1.2}$$

where $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$. The boundary condition is taken as the homogeneous Dirichlet boundary condition without loss of generality. Assume that $\overline{a}_{ij} \in C^1(\Omega_2)$ ($\overline{a}_{ij}$ does not vary with respect to δ) and that σ is bounded and

$$\frac{\sigma}{\delta} \to \alpha \in [0,\infty] \text{ as } \delta \to 0^+. \tag{1.3}$$

Then under some conditions on the initial data φ and the source term f, using energy method, the authors have shown that the weak solution of (1.2) satisfies $u \to w$ in $L^2(\Omega_1 \times (0,T))$ as $\delta \to 0^+$ (in fact, in $C\left([0,T]; L^2(\Omega_1)\right)$; see [4, 6]), where w is the weak solution of

$$\begin{cases} w_t - k\Delta w = f(x,t), & (x,t) \in \Omega_1 \times (0,T), \\ k\dfrac{\partial w}{\partial \nu} + \alpha\nu\cdot\nu_{\overline{A}} w = 0, & (x,t) \in \partial\Omega_1 \times (0,T), \\ w = \varphi(x), & x \in \Omega_1,\ t = 0. \end{cases} \tag{1.4}$$

Here ν is the outward unit normal vector field on $\partial\Omega_1$ and $\nu_{\overline{A}}$ is the conormal vector $\overline{a}_{ij}\nu_j$. If $\alpha = \infty$, the boundary condition is understood as the homogeneous Dirichlet boundary condition. This result implies that the body Ω_1 is perfectly insulated if and only if

$$\frac{\sigma}{\delta} \to 0 \text{ as } \delta \to 0^+.$$

Although thermal barrier coatings have been developed for a variety of applications, the issue of coating failure precluded their wide application in practice [1, 11]. The failure is due to the large stress between the ceramic topcoat and the metallic surface at high temperature. To solve this issue, a technique of using "functionally graded materials"(FGMs) was proposed in the middle 1980s [13]. FGM is meant to replace the sharp interface between two materials with a gradient interface that makes a smooth, gradual transition from one material to the other. This motivates us to study the asymptotic behavior of (1.2), but with Ω being given in Figure 2, where the coating Ω_2 is composed of two layers, the mixed part Ω_3 and the pure ceramic part $\Omega_2 \setminus \Omega_3$. In this paper, we will model FGM coating by assuming that the thermal conductivity is a continuous function of the thickness variable that makes a smooth transition from k (thermal conductivity of the body Ω_1) to σ (thermal conductivity of the ceramic) in the mixed part Ω_3 whose thickness is $\delta_1 \in (0, \delta)$. If $\delta_1 = 0$, then Figure 2 reduces to Figure 1. Therefore, our study here can be viewed as a generalization of [8].

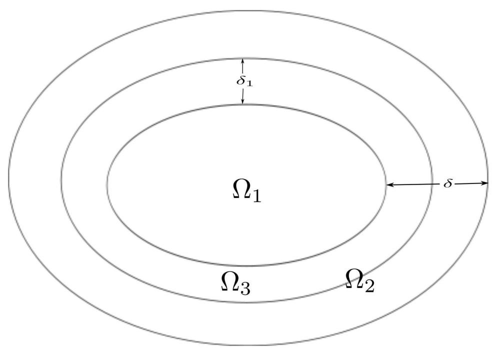


FIGURE 2. $\Omega = \overline{\Omega}_1 \cup \Omega_2$. The coating Ω_2 is uniformly thick with thickness δ and the mixed part Ω_3 has thickness $\delta_1 \in (0, \delta)$.

To give a precise characterization of the thermal tensor $a_{ij}(x)$ on Ω, we need the following parametrization of the coating Ω_2. Define F by

$$\begin{aligned} &F : (p, r) \in \partial\Omega_1 \times [-\delta, \delta] \to (x_1, ..., x_n) \in \mathbb{R}^n, \; n > 1, \\ &(x_1, ..., x_n) = F(p, r) = p + r\nu(p). \end{aligned} \tag{1.5}$$

Then $\Omega_2 = F(\partial\Omega_1 \times (0, \delta))$, and hence Ω_2 is parameterized in (p, r) variables. Moreover, we will always assume $\partial\Omega_1 \in C^2$; then by Lemma 14.16 of [5], $\partial\Omega_2 \in C^2$ if $\delta > 0$ is sufficiently small and the signed distance function $r = r(x)$ is C^2 smooth for x whose distance to $\partial\Omega_1$ is less than δ. Let

$$A(x) = (a_{ij}(x))_{n\times n} = \begin{cases} kI_{n\times n}, & x \in \Omega_1, \\ \sigma_\nu(r) \left(\overline{a}_{ij}(x)\right)_{n\times n}, & x \in \Omega_2, \end{cases} \tag{1.6}$$

where k is a positive constant independent of δ, $I_{n\times n}$ is the identity matrix and $(\overline{a}_{ij})$ is a symmetric positive-definite matrix. $\sigma_\nu(r)$ is a smooth function in $r \in (0, \delta_1)$ satisfying $\sigma_\nu(0) = k$ and $\sigma_\nu(r) \equiv \sigma$ for $r \in [\delta_1, \delta)$, where we have used the coordinates in (1.5). The size of $\sigma_\nu(r)$ measures the thermal conductivity in all directions. Moreover, since physically the mixed part Ω_3 is meant to make a smooth transition from being metallic to being ceramic, it makes sense to assume that $\min(k, \sigma) \leq \sigma_\nu(r) \leq \max(k, \sigma)$ for $r \in (0, \delta_1)$. We also assume that σ is bounded as it is the case in thermal barrier coatings. Roughly speaking, we model the FGM coating by assuming that the material is graded in the normal direction. In fact, our characterization of FGM here is even more general than that in literature under the context of heat transfer, where it is usually assumed $\overline{a}_{ij}$ being a constant matrix; see, for example [1].

In this paper, we study the EBC on $\partial\Omega_1$ that is approximately satisfied by the solution to the heat equation (1.2) with Ω in Figure 2 and $A(x)$ being given by (1.6). We obtain the following result. Let

$$C_\delta = \frac{1}{\int_0^\delta \frac{1}{\sigma_\nu(r)}\,dr}, \tag{1.7}$$

and redefine α as

$$\lim_{\delta\to 0^+} C_\delta = \alpha \in [0,\infty]. \tag{1.8}$$

Then, under appropriate conditions, for any fixed and finite $T > 0$, we have the convergence $u \to w$ in $C\left([0,T]; L^2(\Omega_1)\right)$, where w is the solution of (1.4). If $\alpha = \infty$, then the boundary condition is understood as the homogeneous Dirichlet boundary condition. Note that obviously $C_\delta = \frac{\sigma}{\delta}$ if $\delta_1 = 0$, and therefore our result here covers that of [8].

In this paper, we also study the case of "optimally aligned coating", which was first introduced mathematically in [12]. We say the coating Ω_2 is optimally aligned if $\forall x \in \Omega_2$, the vector $\overrightarrow{px}$ is an eigenvalue of $a_{ij}(x)$, where p is the projection of x onto $\partial\Omega_1$. Note that in this case $a_{ij}(x)$ on Ω_2 is not assumed to be in the form of (1.6). Now we redefine $\sigma_\nu(x) = \sigma_\nu(r)$ to be the eigenvalue of $a_{ij}(x)$ corresponding to the eigenvector $\overrightarrow{px}$ for $x \in \Omega_2$, where $\sigma_\nu(r)$ satisfies all the conditions described above. That is, in the context of optimally aligned coating, we model FGM by assuming that solely the conductivity in the normal direction be in the size of $\sigma_\nu(r)$. Since it measures the thermal conductivity in the normal direction, we call it *normal thermal conductivity*. All other eigenvalues of the thermal tensor $a_{ij}(x)$ are called *tangential thermal conductivities*. Assume σ is bounded and the analog of (1.8),

$$\lim_{\delta\to 0^+} \mu\delta = 0 \quad \text{and} \quad \lim_{\delta\to 0^+} C_\delta = \alpha \in [0,\infty],$$

where μ is an upper bound for all tangential thermal conductivities over Ω_2 and C_δ is given in (1.7). It is shown that the above result still holds with a slight modification. See Theorem 2.4 for details. This implies that the tangential thermal conductivities do not affect the EBC, as long as they are not larger than $o(1/\delta)$. Physically, this indicates that when designing the coating Ω_2, engineers can focus on the normal direction, without worrying about the tangential directions too much. We emphasize that exotic EBCs may appear if the tangential thermal conductivities are larger than $o(1/\delta)$ and refer interested readers to [4] for the study of the linear heat equation in a two-dimensional spatial domain.

EBCs were first formally studied in the classical book of Carslaw and Jaeger [3] in 1959. And then were rigorously investigated by Brezis, Caffarelli and Friedman [2] in 1980 for linear elliptic equations. [7] and [10] gave some new and further developments. [8] discussed the EBCs of the linear heat equation under Dirichlet boundary condition, while [9] studied that for Robin problem of the linear heat equation. In a 2-D spatial domain and in the case of optimally aligned coating, [4] studied some exotic EBCs for both the Dirichlet problem and Neumann problem of linear heat equations. See also [6] for the study of EBCs for a logistic diffusion equation.

This paper is organized as follows. Section 2 is devoted to the study of the asymptotic behavior of u and EBCs via $W_2^{1,1}$ estimates, both in the general case and the optimally aligned case. In Section 3, we consider a concrete example where the material Ω_3 is linearly graded. By explicitly studying the limit of C_δ, from the viewpoint of perfect protection of the body Ω_1, we compare our result with that of [8], through which we are able to see the effect of the mixed part Ω_3. We also provide some numerical simulations in one dimension to verify our analytical results.

In the sequel, we always assume that Ω is as given in Figure 2, $\partial\Omega_1$ is C^2-smooth, the diffusion tensor $A(x) = kI_{n\times n}$ in Ω_1 with k independent of δ, and on Ω, $A \in L^\infty$ is symmetric with its smallest eigenvalue being bounded from below by a positive constant (which may be dependent of δ).

2. The asymptotic behavior of u and EBCs.

2.1. Preliminaries. We first introduce notations for various Sobolev spaces. Let $W_2^{1,0}(Q_T)$ be the Sobolev space consisting of L^2 functions defined on Q_T whose first order weak derivatives in x are also in $L^2(Q_T)$; let $W_2^{1,1}(Q_T)$ be defined similarly but with first order weak derivative in t being in $L^2(Q_T)$ too. Denote by $W_{2,0}^{1,0}(Q_T)$ the closure in $W_2^{1,0}(Q_T)$ of C^∞ functions that vanish near $\overline{S}_T$, and define $W_{2,0}^{1,1}(Q_T)$ similarly. Let $V_{2,0}^{1,0}(Q_T) = W_{2,0}^{1,0}(Q_T) \bigcap C\left([0,T]; L^2(\Omega)\right)$.

Definition 2.1. $u(x,t) \in V_{2,0}^{1,0}(Q_T)$ is called a weak solution of (1.2), if u satisfies $u(\cdot,t) \to \varphi(\cdot)$ in $L^2(\Omega)$ as $t \to 0$, and for any $v \in W_{2,0}^{1,1}(Q_T)$ satisfying $v = 0$ at $t = 0$ and $t = T$, it holds

$$-\int_0^T \int_\Omega u v_t dxdt + \int_0^T \int_\Omega a_{ij}(x) u_{x_i} v_{x_j} dxdt = \int_0^T \int_\Omega f v dxdt. \tag{2.1}$$

For $\varphi \in L^2(\Omega)$ and $f \in L^2(Q_T)$, by Galerkin method, it can be shown that there exists a unique weak solution $u \in V_{2,0}^{1,0}(Q_T)$. Moreover, $u \in W_2^{1,1}((t_0,T)\times\Omega)$ for any $t_0 > 0$ and $u \in C\left((0,T]; H_0^1(\Omega)\right)$.

The following basic energy estimates are already proved in [4]. These estimates will be used frequently in our forthcoming argument.

Lemma 2.2. *Suppose $\varphi \in L^2(\Omega)$ and $f \in L^2(Q_T)$. Then the weak solution $u(x,t)$ of (1.2) satisfies*

$$\begin{aligned}&\max_{0\le t\le T}\int_\Omega u^2(x,t)dx + 2\int_{Q_T} a_{ij}u_{x_i}u_{x_j}\,dxdt\\ &\le 2e^T\left(\int_\Omega \varphi^2\,dx + \int_{Q_T} f^2\,dxdt\right)\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\int_{Q_T} t u_t^2\,dxdt + \max_{0\le t\le T} t\int_\Omega a_{ij}u_{x_i}u_{x_j}\,dx\\ &\le 2\left(1+e^T\right)\left(\int_\Omega \varphi^2\,dx + \int_{Q_T} f^2\,dxdt\right).\end{aligned} \tag{2.3}$$

2.2. **Asymptotics and EBCs of u as $\delta \to 0$.**

Theorem 2.3. *Assume that $A(x)$ is given by (1.6), $\overline{a}_{ij}(x) \in C^1\left(\overline{\Omega}_2\right)$, $\varphi \in L^2(\Omega)$ and $f \in L^2\left(Q_T\right)$ with both functions remaining unchanged as $\delta \to 0^+$. Suppose $\overline{a}_{ij}$ does not vary as δ shrinks; $\sigma_\nu(r)$ is a continuous function in $(0,\delta)$ with $\sigma_\nu(0) = k$ and $\sigma_\nu(r) \equiv \sigma$ for $r \in [\delta_1, \delta)$ and $\min(k,\sigma) \le \sigma_\nu(r) \le \max(k,\sigma)$. Moreover, suppose σ is bounded and $\lim_{\delta\to 0^+} C_\delta = \alpha \in [0,\infty]$, where C_δ is given in (1.7). Then for any fixed and finite $T > 0$, the weak solution $u(x,t)$ of (1.2) satisfies $u(x,t) \to w(x,t)$ strongly in $C\left([0,T]; L^2(\Omega_1)\right)$ as $\delta \to 0^+$, where w is the solution of (1.4). The boundary condition is understood as the homogeneous Dirichlet boundary condition if $\alpha = \infty$.*

Proof. It follows from Lemma 2.2 that $\{u\}_{\delta>0}$ is bounded in the following spaces: $W_2^{1,0}\left(\Omega_1 \times (0,T)\right)$, $W_2^{1,1}(\Omega_1 \times (t_0,T))$ and $C\left([t_0,T]; H^1(\Omega_1)\right)$, for any fixed small $t_0 > 0$. The boundedness in $W_2^{1,0}\left(\Omega_1 \times (0,T)\right)$ implies that after passing to a subsequence, $u \to$ some w weakly in $W_2^{1,0}\left(\Omega_1 \times (0,T)\right)$ as $\delta \to 0^+$. For any fixed $t \in [t_0,T]$, since $\|u(\cdot,t)\|_{H^1(\Omega_1)}$ is bounded and therefore for small $\delta > 0$, $\{u(\cdot,t)\}_{\delta>0}$ is precompact in $L^2(\Omega_1)$. Moreover, in view of that $\int_{Q_T} t u_t^2\, dxdt$ is bounded, we can infer that the functions $\{u\}_{\delta>0} : t \in [t_0,T] \to u(\cdot,t) \in L^2(\Omega_1)$ are equicontinuous. An application of the generalized Arzela-Ascoli theorem gives that after further passing to a subsequence of $\delta \to 0^+$, $u(\cdot,t) \to w(\cdot,t)$ strongly in $C\left([t_0,T]; L^2(\Omega_1)\right)$.

In the following we shall show the convergence of u to w in $C\left([0,T]; L^2(\Omega_1)\right)$. To this aim, we take a sequence $\{\varphi_m\}_{m=1}^{\infty} \subset C_0^\infty(\Omega_1)$ such that $\|\varphi - \varphi_m\|_{L^2(\Omega_1)} \le \frac{1}{m}$. Now decompose u as $u = u_1 + u_2$, where u_1 is the weak solution of

$$\begin{cases} (u_1)_t = \nabla \cdot \left(A(x)\nabla u_1\right), & (x,t) \in Q_T, \\ u_1 = 0, & (x,t) \in S_T, \\ u_1(x,0) = (\varphi - \varphi_m)(x), & x \in \Omega, \end{cases} \tag{2.4}$$

and u_2 is the weak solution of

$$\begin{cases} (u_2)_t = \nabla \cdot (A(x)\nabla u_2) + f, & (x,t) \in Q_T, \\ u_2 = 0, & (x,t) \in S_T, \\ u_2(x,0) = \varphi_m(x), & x \in \Omega. \end{cases} \tag{2.5}$$

Since (2.4) is homogeneous, it holds

$$\|u_1(\cdot,t)\|_{L^2(\Omega)} \le \|\varphi - \varphi_m\|_{L^2(\Omega)} \le \frac{1}{m} + \|\varphi\|_{L^2(\Omega_2)}.$$

Note $\varphi_m \in C_0^\infty(\Omega_1) \subset H_0^1(\Omega)$, multiplying the PDE in (2.5) by $(u_2)_t$, integrating by parts and using the Cauchy-Schwartz inequality, we are led to

$$\begin{aligned} &\int_0^t \int_\Omega \left((u_2)_t\right)^2 dxds + \int_\Omega A(x)\nabla u_2 \cdot \nabla u_2 dx \\ &\le \int_\Omega A(x)\nabla\varphi_m \cdot \nabla\varphi_m dx + \int_0^t \int_\Omega f^2 dxds \\ &\le k\|\nabla\varphi_m\|_{L^2(\Omega_1)}^2 + \|f\|_{L^2(Q_T)}^2. \end{aligned}$$

As a result, for any small $\varepsilon > 0$ and $t \in [0, \varepsilon]$, we have

$$\begin{aligned}
&\|u_2(\cdot,t) - \varphi_m(\cdot)\|^2_{L^2(\Omega)} \\
&= 2\int_0^t \int_\Omega (u_2(x,s) - \varphi_m(x))(u_2)_t(x,s)dxds \\
&\leq 2\left(\int_0^t \int_\Omega (u_2(x,s) - \varphi_m(x))^2 dxds\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega ((u_2))_t^2 \, dxds\right)^{\frac{1}{2}} \\
&\leq 2\sqrt{\varepsilon} \max_{0\leq t\leq\varepsilon} \|u_2(\cdot,t) - \varphi_m(\cdot)\|_{L^2(\Omega)} \left(k\|\nabla\varphi_m\|^2_{L^2(\Omega_1)} + \|f\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}},
\end{aligned}$$

from which it follows

$$\max_{0\leq t\leq\varepsilon} \|u_2(\cdot,t) - \varphi_m(\cdot)\|_{L^2(\Omega)} \leq 2\sqrt{\varepsilon}\left(k\|\nabla\varphi_m\|^2_{L^2(\Omega_1)} + \|f\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}}.$$

The above estimates imply that for $t \in [0, \varepsilon]$,

$$\begin{aligned}
&\|u(\cdot,t) - \varphi(\cdot)\|_{L^2(\Omega_1)} \\
&\leq \|u_1(\cdot,t)\|_{L^2(\Omega_1)} + \|u_2(\cdot,t) - \varphi_m(\cdot)\|_{L^2(\Omega_1)} + \|\varphi_m(\cdot) - \varphi(\cdot)\|_{L^2(\Omega_1)} \\
&\leq \frac{2}{m} + \|\varphi\|_{L^2(\Omega_2)} + 2\sqrt{\varepsilon}\left(k\|\nabla\varphi_m\|^2_{L^2(\Omega_1)} + \|f\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, $\|u(\cdot,t) - \varphi(\cdot)\|_{L^2(\Omega_1)}$ can be made arbitrarily close to 0 for $t \in [0,\varepsilon]$, if ε is close enough to 0. This, together with the fact that $u \to w$ in $C\left([\varepsilon,T]; L^2(\Omega_1)\right)$ implies $u \to w$ in $C\left([0,T]; L^2(\Omega_1)\right)$ as $\delta \to 0^+$ if we define $w(\cdot,0) = \varphi(\cdot)$. Note also we have just shown $w(\cdot,t) \to \varphi(\cdot)$ in $L^2(\Omega_1)$ as $t \to 0$, i.e., w satisfies the initial condition in (1.4). Recall $w \in V_2^{1,0}(Q_T)$.

We now prove that w is a weak solution of (1.4). For any point $p \in \partial\Omega_1$, denote by l the ray in the conormal direction $\nu_{\overline{A}}$ initiated at p. It intersects $\partial\Omega$ at a point q^δ. Let q be projection of q^δ on $\partial\Omega_1$ so that $q^\delta = q + \delta\nu(q)$. Denote by $h(p)$ the distance between p and q^δ. It is shown in [8] that

$$\frac{h(p)}{\delta} \to \frac{|\nu_{\overline{A}}(p)|}{\nu(p)\cdot\nu_{\overline{A}}(p)}, \tag{2.6}$$

uniformly for $p \in \partial\Omega_1$ as $\delta \to 0^+$. We now reparameterize Ω_2 by the following

$$(x_1,\cdots,x_n) = \overline{F}(p,s) = p + s\frac{\nu_{\overline{A}}}{|\nu_{\overline{A}}|}(p), \tag{2.7}$$

with $s \in (0, h(p))$. Recall from [8] that the volume element on Ω_2 at $\overline{F}(p,s)$ is

$$dx_1\cdots dx_n = \left(\frac{\nu\cdot\nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s)\right) dS_p ds, \tag{2.8}$$

where dS_p is the surface element on $\partial\Omega_1$.

Now set

$$\phi(x) = \begin{cases} C_\delta \int_r^\delta \frac{1}{\sigma_\nu(r)}\,dr, & r \in [0,\delta],\ p \in \partial\Omega_1, \\ 1, & \text{otherwise}, \end{cases} \tag{2.9}$$

where we have used the coordinates introduced in (1.5). Observe that ϕ is continuous and $0 \leq \phi \leq 1$. For any $\zeta(x,t) \in C^1\left(\overline{\Omega}_1 \times [0,T]\right)$ satisfying $\zeta(x,t) = 0$ at $t = 0$ and $t = T$, we extend it along $\nu_{\overline{A}}(p)$ to Ω by setting $\zeta(p,s,t) = \zeta(p,0,t)$ for every

$p \in \partial\Omega_1$, then it is easy to verify that $\zeta\phi \in W_{2,0}^{1,1}(Q_T)$. Taking $v = \zeta\phi$ as a test function in (2.1), we obtain

$$-\int_0^T\int_{\Omega_1} u\zeta_t dxdt - \int_0^T\int_{\Omega_2} u\phi\zeta_t dxdt + k\int_0^T\int_{\Omega_1} \nabla u\cdot\nabla\zeta dxdt$$
$$+\int_0^T\int_{\Omega_2} \sigma_\nu \overline{a}_{ij}(x)u_{x_i}\left(\zeta\phi\right)_{x_j}\ dxdt$$
$$=\int_0^T\int_{\Omega_1} f\zeta dxdt + \int_0^T\int_{\Omega_2} f\zeta\phi dxdt. \tag{2.10}$$

By Hölder inequality and Lemma 2.2, we have

$$\left|\int_0^T\int_{\Omega_2} u\phi\zeta_t dxdt\right| \le \left(\int_0^T\int_{\Omega_2} u^2 dxdt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\Omega_2} \zeta_t^2 dxdt\right)^{\frac{1}{2}} = O\left(\sqrt{\delta}\right),$$

and

$$\left|\int_0^T\int_{\Omega_2} \sigma_\nu\phi\overline{a}_{ij}(x)u_{x_i}\zeta_{x_j} dxdt\right|$$
$$\le \left(\int_0^T\int_{\Omega_2} \sigma_\nu \overline{a}_{ij}(x)u_{x_i}u_{x_j}\ dxdt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\Omega_2} \sigma_\nu \overline{a}_{ij}(x)\zeta_{x_i}\zeta_{x_j}\ dxdt\right)^{\frac{1}{2}}$$
$$\le C\left(\int_0^T\int_{\Omega_2} \sigma_\nu \overline{a}_{ij}(x)\zeta_{x_i}\zeta_{x_j}\ dxdt\right)^{\frac{1}{2}}$$
$$\le C\left(\int_0^\delta \sigma_\nu(r)\ dr\right)^{\frac{1}{2}}$$
$$\le O\left(\sqrt{\delta}\right),$$

where we have used the assumption $\sigma_\nu \le \max(k,\sigma) \le O(1)$ since σ is bounded. Moreover,

$$\left|\int_0^T\int_{\Omega_2} f\zeta\phi dxdt\right| \le \left(\int_0^T\int_{\Omega_2} f^2\ dxdt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\Omega_2} \zeta^2\ dxdt\right)^{\frac{1}{2}} = O\left(\sqrt{\delta}\right).$$

By virtue of (2.9), for $x \in \Omega_2$, we have $\nabla\phi(x) = \frac{\partial\phi}{\partial r}\nu(p) = -\frac{C_\delta}{\sigma_\nu(r)}\nu(p)$. Therefore

$$\int_0^T\int_{\Omega_2} \sigma_\nu\zeta\overline{a}_{ij}(x)u_{x_i}\phi_{x_j} dxdt$$
$$= -C_\delta\int_0^T\int_{\Omega_2} \zeta\overline{a}_{ij}(x)u_{x_i}\nu_j(p)dxdt$$
$$= -C_\delta\int_0^T\int_{\partial\Omega_1}\int_0^{h(p)} \left[\overline{a}_{ij}(p,s) - \overline{a}_{ij}(p,0)\right]u_{x_i}\nu_j\zeta\left(\frac{\nu\cdot\nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s)\right)dsdS_pdt$$
$$-C_\delta\int_0^T\int_{\partial\Omega_1}\int_0^{h(p)} O(s)\overline{a}_{ij}(p,0)u_{x_i}\nu_j\zeta dsdS_pdt$$

$$-C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \overline{a}_{ij}(p,0) u_{x_i} \nu_j \zeta \frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} ds dS_p dt$$
$$\triangleq \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{2.11}$$

with

$$\mathrm{I} = -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \left[\overline{a}_{ij}(p,s) - \overline{a}_{ij}(p,0)\right] u_{x_i} \nu_j \zeta \left(\frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s) \right) ds dS_p dt,$$
$$\mathrm{II} = -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} O(s) \overline{a}_{ij}(p,0) u_{x_i} \nu_j \zeta ds dS_p dt,$$

and

$$\mathrm{III} = -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \overline{a}_{ij}(p,0) u_{x_i} \nu_j \zeta \frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} ds dS_p dt.$$

We then estimate each term. Since $\min(k,\sigma) \leq \sigma_\nu(r) \leq \max(k,\sigma)$ and σ is bounded, it follows from (1.7) that

$$\frac{\min(k,\sigma)}{\delta} \leq C_\delta \leq \frac{\max(k,\sigma)}{\delta} = O\left(\frac{1}{\delta}\right). \tag{2.12}$$

Owing to $|\overline{a}_{ij}(p,s) - \overline{a}_{ij}(p,0)| \leq O(s)$, (2.6), and Lemma 2.2, we have

$$\begin{aligned}
|\mathrm{I}| &\leq C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} O(s) \left| u_{x_i} \nu_j \zeta \right| \left(\frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s) \right) ds dS_p dt \\
&\leq C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} O\left(h(p)\right) |\nabla u \zeta| \left(\frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s) \right) ds dS_p dt \\
&\leq O(\delta)\, C_\delta \int_0^T \int_{\Omega_2} \frac{h(p)}{\delta} |\nabla u \zeta| dx dt \\
&\leq O(\delta)\, C_\delta \left(\int_0^T \int_{\Omega_2} \sigma_\nu \overline{a}_{ij}(x) u_{x_i} u_{x_j}\, dx dt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Omega_2} \frac{1}{\sigma_\nu(r)} dx dt \right)^{\frac{1}{2}} \\
&\leq \frac{O(\delta)}{\left(\int_0^\delta \frac{1}{\sigma_\nu(r)}\, dr \right)^{\frac{1}{2}}} \\
&\leq O(\delta) \sqrt{C_\delta}, \\
&\leq O\left(\sqrt{\delta}\right),
\end{aligned}$$

where we have used (1.7) and (2.12). Moreover,

$$\begin{aligned}
|\mathrm{II}| &\leq C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} O\left(h(p)\right) \left| \overline{a}_{ij}(p,0) u_{x_i} \nu_j \zeta \right| ds dS_p dt \\
&\leq O(\delta)\, C_\delta \int_0^T \int_{\Omega_2} \frac{O\left(h(p)\right)}{\delta} |\nabla u \zeta| dx dt \\
&\leq O\left(\sqrt{\delta}\right),
\end{aligned}$$

as above. Next, observing $u_{x_i}\overline{a}_{ij}(p,0)\nu_j = \frac{\partial u}{\partial \nu_{\overline{A}}}$ and $\zeta(p,s,t) = \zeta(p,0,t)$, we are led to

$$\begin{aligned}
\mathrm{III} &= -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \overline{a}_{ij}(p,0) u_{x_i} \nu_j \zeta \frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} ds dS_p dt \\
&= -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \frac{\partial u}{\partial \nu_{\overline{A}}} \zeta(p,0,t) \frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} ds dS_p dt \\
&= -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \frac{\partial u}{\partial s} \zeta(p,0,t) \nu \cdot \nu_{\overline{A}} ds dS_p dt \\
&= C_\delta \int_0^T \int_{\partial\Omega_1} \zeta \nu \cdot \nu_{\overline{A}} u(p,t) dS_p dt.
\end{aligned}$$

By (2.10) and the above estimates, we conclude that

$$\begin{aligned}
&- \int_0^T \int_{\Omega_1} u \zeta_t dx dt + k \int_0^T \int_{\Omega_1} \nabla u \cdot \nabla \zeta dx dt + C_\delta \int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} u(p,t) \zeta dS_p dt \\
&= \int_0^T \int_{\Omega_1} f \zeta dx dt + O\left(\sqrt{\delta}\right). \qquad (2.13)
\end{aligned}$$

Let us recall that $u(x,t) \to w(x,t)$ weakly in $W_2^{1,0}(\Omega_1 \times (0,T))$ and strongly in $C\left([0,T]; L^2(\Omega_1)\right)$.

If $\lim_{\delta\to 0^+} C_\delta = \alpha \in [0,\infty)$, observe that the linear functional

$$u \in W_2^{1,0}(\Omega_1 \times (0,T)) \mapsto \int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} u(p,t) \zeta dS_p dt$$

is bounded, then as $\delta \to 0^+$, it holds

$$\int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} u(p,t) \zeta dS_p dt \to \int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} w(p,t) \zeta dS_p dt. \qquad (2.14)$$

This, in conjunction with (2.13), leads to

$$\begin{aligned}
&- \int_0^T \int_{\Omega_1} w \zeta_t dx dt + k \int_0^T \int_{\Omega_1} \nabla w \cdot \nabla \zeta dx dt + \alpha \int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} w \zeta dS_p dt \\
&= \int_0^T \int_{\Omega_1} f \zeta dx dt. \qquad (2.15)
\end{aligned}$$

If $C_\delta \to \infty$ as $\delta \to 0^+$, dividing (2.13) by C_δ, we arrive at

$$\int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} u(p,t) \zeta dS_p dt \to 0.$$

Thanks to (2.14) again, we have

$$\int_0^T \int_{\partial\Omega_1} \nu \cdot \nu_{\overline{A}} w(p,t) \zeta dS_p dt = 0.$$

The arbitrariness of ζ implies that the trace of w on $\partial\Omega_1 \times (0,T)$ is zero. Now, let ζ be arbitrary with compact support in $\Omega_1 \times (0,T)$, then the last term of (2.13) is zero since it comes from the integrals on Ω_2 involving ζ. Therefore, we obtain (2.15) without $\alpha-$term, i.e., w is the weak solution to (1.4) subject to homogeneous Dirichlet boundary condition. □

2.3. **Optimally aligned coating.** In this subsection we prove that in the case of optimally aligned coating, Theorem 2.3 still holds with a slight modification. Recall that in this case, we do not assume that the diffusion tensor $A(x)$ is in the form of (1.6) on Ω_2.

Theorem 2.4. *Assume that $\varphi \in L^2(\Omega)$ and $f \in L^2(Q_T)$ with both functions remaining unchanged as $\delta \to 0^+$. Suppose that the coating Ω_2 is optimally aligned, and that the eigenvalue $\sigma_\nu(x) = \sigma_\nu(r)$ of $A(x)$ on Ω_2 in the normal direction of $\overrightarrow{px}$ satisfies all the conditions mentioned in Theorem 2.3. Suppose $\lim_{\delta\to 0^+} \mu\delta = 0$, where μ is an upper bound for all other eigenvalues of $a_{ij}(x)$ on Ω_2. Assume σ is bounded and $\lim_{\delta\to 0^+} C_\delta = \alpha \in [0,\infty]$, where C_δ is given in (1.7). Then for any fixed finite $T > 0$, the weak solution $u(x,t)$ of (1.2) satisfies $u(x,t) \to w(x,t)$ strongly in $C\left([0,T]; L^2(\Omega_1)\right)$ as $\delta \to 0^+$, where w is the solution of*

$$\begin{cases} w_t = k\Delta w + f, & x \in \Omega_1,\ t \in (0,T), \\ k\dfrac{\partial w}{\partial \nu} + \alpha w = 0, & x \in \partial\Omega_1,\ t \in (0,T), \\ w = \varphi(x), & x \in \Omega_1,\ t = 0. \end{cases} \tag{2.16}$$

The boundary condition is understood as homogeneous Dirichlet boundary condition if $\alpha = \infty$.

Proof. The proof here is a modification of that of Theorem 2.3 without reparametrizing Ω_2. Here we only point out some modifications.

Recall the curvilinear coordinate system (1.5). Similar to [8], it can be checked that the volume element on Ω_2 at $F(p,r)$ is

$$dx_1 \cdots dx_n = (1 + O(r))\, dS_p dr,$$

where dS_p is the surface element on $\partial\Omega_1$.

For any $\zeta(x,t) \in C^1\left(\overline{\Omega}_1 \times [0,T]\right)$ satisfying $\zeta(x,t) = 0$ at $t = 0$ and $t = T$, extend ζ along $\nu(p)$ to Ω by setting $\zeta(p,r,t) = \zeta(p,0,t)$ for every $p \in \partial\Omega_1$, then it is easy to verify that $\zeta\phi \in W^{1,1}_{2,0}(Q_T)$, where ϕ is given in (2.9). We now take the same test function as before. Notice

$$\begin{aligned} &\left| \int_0^T \int_{\Omega_2} \phi\, a_{ij}(x) u_{x_i} \zeta_{x_j}\, dxdt \right| \\ &\le \left(\int_0^T \int_{\Omega_2} a_{ij}(x) u_{x_i} u_{x_j}\, dxdt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Omega_2} a_{ij}(x) \zeta_{x_i} \zeta_{x_j}\, dxdt \right)^{\frac{1}{2}} \\ &\le C\sqrt{\max(\mu,k,\sigma)} \left(\int_0^T \int_{\Omega_2} |\nabla\zeta|^2 dxdt \right)^{\frac{1}{2}} \\ &\le O\left(\sqrt{\mu\delta}\right) + O\left(\sqrt{\delta}\right), \end{aligned}$$

where we have used $\sigma_\nu(r) \le \max(k,\sigma) \le O(1)$. Moreover, observing that $\nabla\phi(x) = \frac{\partial\phi}{\partial r}\, \nu(p) = -\frac{C_\delta}{\sigma_\nu(r)}\, \nu(p)$ is an eigenvector of $a_{ij}(x)$ corresponding to σ_ν and $a_{ij}(x)\nu_j =$

$\sigma_\nu(r)\nu_i$, we are led to,

$$
\begin{aligned}
&\int_0^T \int_{\Omega_2} \zeta a_{ij}(x) u_{x_i} \phi_{x_j}\, dxdt \\
&= \int_0^T \int_{\Omega_2} \zeta a_{ij}(x) u_{x_i} \left(-\frac{C_\delta}{\sigma_\nu(r)}\right) \nu_j(p)\, dxdt \\
&= -C_\delta \int_0^T \int_{\Omega_2} \zeta \nabla u \cdot \nu(p)\, dxdt \\
&= -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^\delta \zeta(p,t) \frac{\partial u}{\partial r} \left(1 + O(r)\right)\, drdS_pdt \\
&\triangleq \mathrm{I} + \mathrm{II},
\end{aligned} \tag{2.17}
$$

with

$$
\begin{aligned}
\mathrm{I} &= -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^\delta \zeta(p,t) \frac{\partial u}{\partial r}\, drdS_pdt, \\
\mathrm{II} &= -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^\delta \zeta(p,t) \frac{\partial u}{\partial r} O(r)\, drdS_pdt.
\end{aligned}
$$

The boundary condition of u implies

$$
\mathrm{I} = C_\delta \int_0^T \int_{\partial\Omega_1} \zeta(p,t) u(p,t)\, dS_pdt.
$$

We estimate the second term as follows.

$$
\begin{aligned}
|\mathrm{II}| &= \left| -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^\delta O(r) \zeta(p,t) u_{x_i} \nu_i \, drdS_pdt \right| \\
&= \left| -C_\delta \int_0^T \int_{\partial\Omega_1} \int_0^\delta O(r) \zeta(p,t) u_{x_i} \frac{a_{ij}\nu_j}{\sigma_\nu}\, drdS_pdt \right| \\
&\le O(\delta)\, C_\delta \left(\int_0^T \int_{\partial\Omega_1} \int_0^\delta a_{ij} u_{x_i} u_{x_j} \right)^{\frac{1}{2}} \left(\int_0^T \int_{\partial\Omega_1} \int_0^\delta \frac{1}{\sigma_\nu^2} a_{ij} \nu_i \nu_j \right)^{\frac{1}{2}} \\
&\le O(\delta)\, C_\delta \left(\int_0^T \int_{\Omega_2} a_{ij} u_{x_i} u_{x_j}\, dxdt \right)^{\frac{1}{2}} \left(\int_0^\delta \frac{1}{\sigma_\nu}\, dr \right)^{\frac{1}{2}} \\
&\le O(\delta) \frac{1}{\left(\int_0^\delta \frac{1}{\sigma_\nu}\, dr \right)^{\frac{1}{2}}} \\
&= O(\delta) \sqrt{C_\delta} \\
&= O\left(\sqrt{\delta}\right),
\end{aligned}
$$

where we have used (2.12). The rest of modifications is obvious. $\square$

3. **An example: Linearly graded material.** In this section, we assume that the mixed layer Ω_3 is composed of linearly graded material, i.e., for $r \in (0, \delta_1)$, $\sigma_\nu(r)$ is a linear function satisfying $\sigma_\nu(0) = k$ and $\sigma_\nu(\delta_1) = \sigma$. We then investigate the limit of C_δ. Recall that σ is bounded and that $\lim_{\delta\to 0^+} C_\delta = 0$ is desirable for perfect insulation of the body. In contrast with [8], we now incorporate a layer Ω_3 of thickness δ_1. Therefore, besides the scaling relationship between σ and δ, we naturally expect δ_1 to play a role.

3.1. **Theoretical discussion.** It is straightforward to verify that

$$C_\delta = \frac{1}{\frac{\delta-\delta_1}{\sigma} - \frac{\delta_1}{k-\sigma}\ln\frac{\sigma}{k}} = \frac{1}{\frac{\delta}{\sigma} - \frac{\delta_1}{\sigma}\left(\frac{\sigma\ln\sigma}{k-\sigma}+1\right) + \frac{\delta_1}{k-\sigma}\ln k}. \tag{3.1}$$

Also observe

$$\frac{\min(k,\sigma)}{\delta} \le C_\delta \le \frac{\sigma}{\delta-\delta_1}. \tag{3.2}$$

Case (1). $\dfrac{\sigma}{\delta-\delta_1} \to 0$. Obviously (3.2) gives $C_\delta \to 0$.

Case (2). $\dfrac{\sigma}{\delta-\delta_1} \to \alpha \in (0,\infty)$. Note that this implies $\sigma \to 0$. Then by (3.1), it holds

$$\lim_{\delta\to 0^+} C_\delta = \lim_{\delta\to 0^+} \frac{1}{\frac{\delta-\delta_1}{\sigma} - \frac{\delta_1}{k-\sigma}\ln\frac{\sigma}{k}} = \lim_{\delta\to 0^+} \frac{1}{\frac{1}{\alpha} - \frac{\delta_1\ln\sigma}{k}}.$$

As a result, $C_\delta \to 0$ if and only if $\delta_1 \ln\sigma \to -\infty$. (3.2) also implies that we have either Neumann or Robin EBC in this case.

Case (3). $\dfrac{\sigma}{\delta-\delta_1} \to \infty. \Longleftrightarrow \dfrac{\delta-\delta_1}{\sigma} \to 0. \Longrightarrow \lim\limits_{\delta\to 0^+}\dfrac{\delta}{\sigma} = \lim\limits_{\delta\to 0^+}\dfrac{\delta_1}{\sigma}.$

It follows from (3.1) that

$$\lim_{\delta\to 0^+} C_\delta = \lim_{\delta\to 0^+}\left(\frac{\sigma-k}{\delta_1\ln\frac{\sigma}{k}}\right) = \lim_{\delta\to 0^+}\left(\frac{\sigma-k}{\ln\sigma-\ln k}\cdot\frac{1}{\delta_1}\right). \tag{3.3}$$

We then consider the following three subcases:

$$\text{(i) } \frac{\delta}{\sigma} \to 0; \qquad \text{(ii) } \frac{\delta}{\sigma} \to \beta \in (0,\infty); \qquad \text{(iii) } \frac{\delta}{\sigma} \to \infty.$$

Case (3i). If $\sigma \to 0$, then, by (3.1), $C_\delta \to \infty$ since

$$\frac{\delta}{\sigma} - \frac{\delta_1}{\sigma}\left(\frac{\sigma\ln\sigma}{k-\sigma}+1\right) + \frac{\delta_1}{k-\sigma}\ln k \to 0.$$

Now if $\sigma \to \gamma \in (0,\infty)$, by making use of (3.3) and observing that

$$\lim_{\delta\to 0^+}\frac{\sigma-k}{\ln\sigma-\ln k} > 0$$

by the Mean Value Theorem, we again obtain $C_\delta \to \infty$.

Case (3ii). Note that we have $\sigma \to 0$. Obviously

$$\frac{\delta}{\sigma} - \frac{\delta_1}{\sigma}\left(\frac{\sigma\ln\sigma}{k-\sigma}+1\right) + \frac{\delta_1}{k-\sigma}\ln k \to 0.$$

Consequently, $C_\delta \to \infty$.

Case (3iii). We still have $\sigma \to 0$. By (3.3),

$$\lim_{\delta\to 0^+} C_\delta = \lim_{\delta\to 0^+}\left(-\frac{k}{\delta_1 \ln\sigma}\right).$$

Therefore, $C_\delta \to 0$ if and only if $\delta_1 \ln\sigma \to -\infty$.

Now recall from [8] that the body Ω_1 is perfectly insulated if and only if

$$\lim_{\delta\to 0^+}\frac{\sigma}{\delta} = 0.$$

While in our situation here, in the case of linearly graded material, we have the homogeneous Neumann EBC if (i) $\lim_{\delta\to 0^+}\frac{\sigma}{\delta-\delta_1} = 0$ or (ii) $\lim_{\delta\to 0^+}\frac{\sigma}{\delta-\delta_1} \to \alpha \in (0,\infty)$ and $\lim_{\delta\to 0^+}\delta_1\ln\sigma = -\infty$, or (iii) $\lim_{\delta\to 0^+}\frac{\sigma}{\delta-\delta_1} = \infty$, $\lim_{\delta\to 0^+}\frac{\delta}{\sigma} = \infty$ and $\lim_{\delta\to 0^+}\delta_1\ln\sigma = -\infty$. Our result here clearly reveals the effect of FGM coating. FGM diminishes the stress tensor between different materials and therefore increases the possibility of coating success. Moreover, unlike the situation in [8], even if $\lim_{\delta\to 0^+}\frac{\sigma}{\delta-\delta_1} = 0$ is not satisfied, it is still possible to perfectly protect the body Ω_1 by adjusting the thickness δ_1 of the mixed layer Ω_3 so that $\delta_1\ln\sigma \to -\infty$. This apparently gives engineers more options when designing the coating.

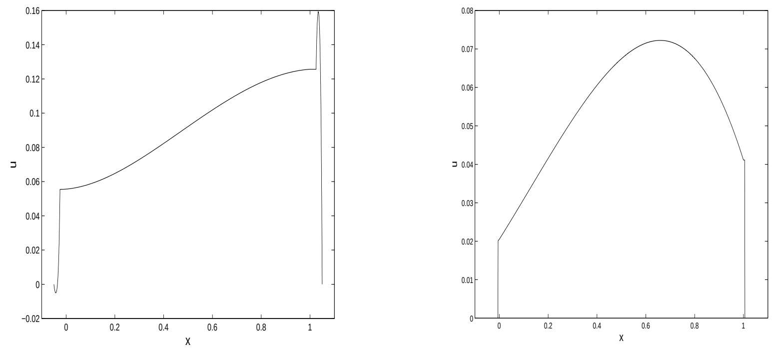
3.2. **Numerical simulation.** This subsection is devoted to the numerical simulation of our analytical results presented above in the one dimensional case. We demonstrate numerically that w with appropriate EBC is a good approximation of u on the body. In all simulations, we take $\Omega_1 = (0,1)$, $\Omega = (-\delta, 1+\delta)$, $k = 1$, $f(x,t) = \sin x$ and the initial condition is taken to be zero. Since we are considering the case of linearly graded material, the thermal conductivity is given by

$$A(x) = \begin{cases} \sigma, & x \in (-\delta,-\delta_1) \text{ or } x \in (1+\delta_1, 1+\delta),\\ \frac{1-\sigma}{\delta_1}x + 1, & x \in (-\delta_1, 0),\\ 1, & x \in (0,1),\\ -\frac{1-\sigma}{\delta_1}(x-1)+1, & x \in (1, 1+\delta_1). \end{cases}$$

Figure 3 illustrates the numerical solution u of problem (1.2) on the entire domain $(-\delta, 1+\delta)$. The finite-difference based Matlab PDE solver is implemented. Due to the small thickness of the coating, we have to take very fine spatial mesh size. Here we take $\Delta x = 2\times 10^{-4}$ and the time step size is $\Delta t = 0.02$. Numerical simulation clearly shows the unpleasant behavior of u around the boundary of the physical domain, again due to the small scales involved in the PDE.

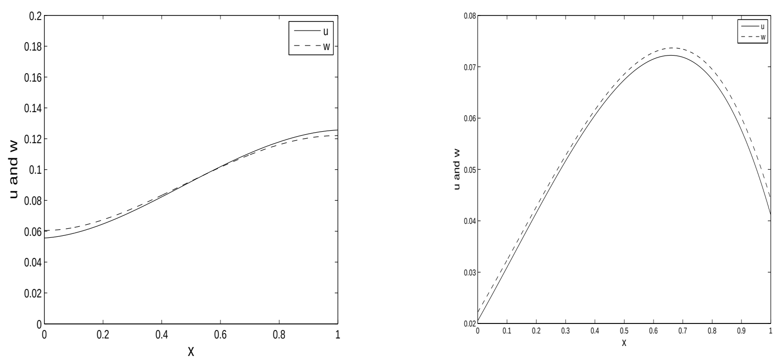
In Figure 4, we approximate the solution u of (1.2) by the solution w of (1.4) subject to EBC. Corresponding to Figure 3 (a), we first take $\delta = 0.05$, $\delta_1 = 0.025$, and $\sigma = 0.00025$. According to (3.1), under these parameter values, we have $C_\delta = 0.01$. Since C_δ is close to 0, we can think of that w is approximately subordinate to Neumann EBC. We now simply solve (1.4) with boundary condition $\frac{\partial w}{\partial \nu} + 0.01w = 0$ (or with Neumann boundary condition, which gives essentially the same result as in Figure 4 (a)). Similarly, by taking the parameter values the same as in Figure 3 (b), one finds that $C_\delta = 4.41$ and then w is subject to the Robin EBC $\frac{\partial w}{\partial \nu} + 4.41w = 0$. Figure 4 illustrates the comparison of u and w on $(0,1)$ under both two scenarios.

The above numerical illustrations evidently show the computational advantage of EBCs: To understand the temperature distribution inside the body, we don't necessarily have to solve (1.2), it is enough to solve the limiting problem (1.4)



(a) (b)

FIGURE 3. Numerical solutions to (1.2) on the entire domain with small scales. (a)-(b) plot the solutions at $t = 0.2$. The parameter values are: (a) $\delta = 0.05$, $\delta_1 = 0.025$, $\sigma = 0.00025$, and (b) $\delta = 0.006$, $\delta_1 = 0.005$, $\sigma = 0.005$.



(a) (b)

FIGURE 4. Comparison of u and w on the body $(0, 1)$, at time $t = 0.2$. (a) and (b) are meant to compare u and w subject to Neumann and Robin EBC, respectively. The parameter values are the same as in Figure 3.

subject to EBCs. Moreover, analytically it is hard to see the effect of the coating from (1.2), but this effect is clearly revealed by our study of EBCs.

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