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Large-time Behavior of Solutions to a Chemotaxis Model in R^2

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LARGE-TIME BEHAVIOR OF SOLUTIONS TO A CHEMOTAXIS MODEL IN $\mathbb{R}^2$

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Contents

1. Introduction

In this lecture we study the large-time behavior of the non-negative solutions of the following Cauchy problem in $\mathbb{R}^2$:

$$\text{(KS)} \qquad \begin{cases} \partial_t u = \Delta u - \nabla\cdot(u(\nabla N * u)), & t>0,\ x\in\mathbb{R}^2, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^2, \end{cases}$$

where $N(x)$ is the Newtonian potential

$$N(x) = \frac{1}{2\pi}\log\frac{1}{|x|}, \qquad \nabla N(x) = -\frac{1}{2\pi}\frac{x}{|x|^2}, \qquad x\in\mathbb{R}^2\setminus\{0\},$$

and $\nabla N * u$ stands for the convolution product

$$(\nabla N * u)(t,x) := \int_{\mathbb{R}^2} \nabla N(x-y)\, u(t,y)\, dy.$$

The problem (KS) comes in a natural way from the Cauchy problem for the nonlinear parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$:

$$\text{(KS)}_\psi \qquad \begin{cases} \partial_t u = \Delta u - \nabla\cdot(u\nabla\psi), & t>0,\ x\in\mathbb{R}^2, \\ -\Delta\psi = u, & t>0,\ x\in\mathbb{R}^2, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^2, \end{cases}$$

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where ψ is specified as

$$\psi(t,x) := (N * u)(t,x) = \int_{\mathbb{R}^2} N(x-y)\, u(t,y)\, dy.$$

The underlying system is a simplified version of a chemotaxis system derived from the original Keller–Segel model [29] (see also Childress–Percus [17]):

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), & t > 0,\ x \in \mathbb{R}^2, \\ \varepsilon \partial_t \psi = \Delta \psi - a\psi + u, & t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ and $a \geq 0$. $(\mathrm{KS})_\psi$ is derived formally by letting $\varepsilon \to 0$ and $a = 0$ in (1.1). $(\mathrm{KS})_\psi$ can be as well regarded as a model of self-attracting particles in $\mathbb{R}^2$ (see Biler–Nadzieja [9] and Wolansky [50]).

Some fundamental properties of the nonnegative solutions to (KS) are the *conservation of the total mass*

$$\int_{\mathbb{R}^2} u(t,x)\, dx = \int_{\mathbb{R}^2} u_0(x)\, dx, \qquad t > 0, \tag{1.2}$$

the *conservation of the center of mass*

$$\int_{\mathbb{R}^2} x u(t,x)\, dx = \int_{\mathbb{R}^2} x u_0(x)\, dx, \qquad t > 0, \tag{1.3}$$

and the *second moment identity*

$$\int_{\mathbb{R}^2} |x|^2 u(t,x)\, dx = \int_{\mathbb{R}^2} |x|^2 u_0(x)\, dx + 4M\Big(1 - \frac{M}{8\pi}\Big)\, t, \qquad t > 0, \tag{1.4}$$

where

$$M := \int_{\mathbb{R}^2} u_0\, dx.$$

The reader can find a rigorous proof of these properties in Blanchet et al. [11, 12].

In model (KS) the total mass, M, of any nonnegative initial data $u_0 \in L^1(\mathbb{R}^2)$ is a significant parameter, because the global existence and large-time behavior of the nonnegative solutions to (KS) heavily depend on M. Precisely, the following features are well documented in the literature.

(i) The *subcritical case* $M < 8\pi$. The nonnegative solution of (KS) is globally defined in time. See [8] for a radially symmetric solution with respect to the space variable, [12] for a weak solution and [36] for a mild solution. We remark that the uniqueness of a (nonnegative) weak solution seems to be still open. In this lecture we consider mild solutions to (KS) for which uniqueness holds.

Equation $\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi)$ is invariant under the similarity transformations $u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x)$ $(\lambda > 0)$, and in the subcritical case, a (forward) self-similar solution of (KS) appears. Given $M > 0$, consider a (forward) self-similar solution U_M of (KS) with $\int_{\mathbb{R}^2} U_M(t,x)\,dx = M$ such that

$$U_M(t,x) = \frac{1}{t}\Phi\Big(\frac{x}{\sqrt{t}}\Big),$$

where

$$\Phi \geq 0 \text{ in } \mathbb{R}^2, \quad \Phi \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

It was shown in [8, 41] that Φ is radially symmetric at a point in $\mathbb{R}^2$, and Φ exists if and only if $0 < M < 8\pi$. The uniqueness of Φ up to the translation of the space variable holds for each $0 < M < 8\pi$.

Given $M = \int_{\mathbb{R}^2} u_0\,dx \in (0, 8\pi)$, let U_M be the radially symmetric self-similar solution at the origin with $\int_{\mathbb{R}^2} U_M(t,x)\,dx = M$. Then the nonnegative solution u converges to U_M as $t \to \infty$. See [8] for the convergence of the mass function $\int_{|x|\leq r} u(t,x)\,dx$ of the radial solution u to $\int_{|x|\leq r} U_M(t,x)\,dx$. For the nonradial solution u, it holds that

$$\lim_{t\to\infty} t^{1-1/p}\|u(t) - U_M(t)\|_{L^p} = 0 \quad (1 \leq p \leq \infty).$$

See [12] for $p = 1$ and [37] for all $1 \leq p \leq \infty$.

(ii) The *supercritical case* $M > 8\pi$. The solution u may blow up in finite time, namely

$$\limsup_{t\to T} \|u(t)\|_{L^\infty} = \infty \quad \text{for a finite time } T > 0.$$

See [9, 12] under the assumption $|x|^2 u_0 \in L^1(\mathbb{R}^2)$ and [30] under $(1 + |x|^2)^{s/2} u_0 \in L^2(\mathbb{R}^2)$ for some $s > 1$.

The number of blowup points of the finite-time blowup solution u is finite and at the blowup time T,

$$\lim_{t\to T} u(t,\cdot)dx = \sum_{x_0\in\mathcal{B}} 8\pi\delta_{x_0}(x) + f(x)dx \quad \text{in the sense of measure,} \tag{1.5}$$

where $\mathcal{B}$ is the set of all blowup points of u, δ_{x_0} is the Dirac distribution at x_0 and f is a nonnegative function in $L^1(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \mathcal{B})$ (see [45, 48]).

See Herrero-Velazquez [23] for a detailed asymptotic profile for the radial solution blowing up in finite time.

(iii) The *critical case* $M = 8\pi$. The problem of ascertaining the dynamics of (KS) is fraught with difficulties. For a nonnegative initial data $u_0 \in L^1(\mathbb{R}^2)$ satisfying $u_0 \log u_0$, $|x|^2 u_0 \in L^1(\mathbb{R}^2)$, the solution grows up at the infinite time and goes to $8\pi\delta_{x_0}$ as $t \to \infty$ (see [11, 44]), where x_0 is the center of mass of u_0

$$x_0 := \frac{1}{M}\int_{\mathbb{R}^2} x u_0(x)\,dx.$$

On the other hand, (KS) admits the stationary solutions

$$w_{b,x_0}(x) = \frac{8b}{(|x - x_0|^2 + b)^2}, \quad b > 0,\ x_0 \in \mathbb{R}^2,$$

which satisfy

$$\int_{\mathbb{R}^2} w_{b,x_0}(x)\,dx = 8\pi \quad \text{and} \quad \int_{\mathbb{R}^2} |x| w_{b,x_0}(x)\,dx < \infty,$$

though the second moment satisfies

$$\int_{\mathbb{R}^2} |x|^2 w_{b,x_0}(x)\,dx = \infty.$$

For some choices of u_0 the solution approximates a stationary solution as $t \to \infty$ (see [8, 10, 27]) and for some other choices of u_0 the omega limit set of u_0 with respect to L^∞-topology contains two different stationary solutions $w_{a,0}$ and $w_{b,0}$ (see [40]).

Some related results regarding to (KS) as a model for chemotaxis were found in [6, 7, 23, 25, 32, 33, 35, 41, 45, 47] as well in [24, 48], while the results of [5, 6] and the references therein regarded to (KS) as a model of self-attracting particles.

The main goal of this lecture is providing a very general condition on $u_0 \geq 0$ in the critical case $M = 8\pi$ so that the solution of (KS) exists globally in time and approximates a stationary solution as $t \to \infty$. This lecture note is written based on the papers [36, 27].

2. Local existence, uniqueness and regularity of mild solutions

We begin with introducing some notations and recall some fundamentals.

Throughout this lecture, for every $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$ stands for the usual Lebesgue space on $\mathbb{R}^d$ with norm $\|\cdot\|_{L^p}$. In case $d = 2$, we denote

$$L^p := L^p(\mathbb{R}^2), \qquad \|\cdot\|_p := \|\cdot\|_{L^p},$$

for simplicity. Moreover, for a given subset $Q \subset \mathbb{R}^d$ and a Banach space X, we denote by $C(Q; X)$ the set of all continuous functions from Q to X. Similarly, $BC(Q; X)$ denotes the set of all bounded continuous functions. When $X = \mathbb{R}$, we simply denote

$$C(Q) := C(Q; \mathbb{R}), \qquad BC(Q) := BC(Q; \mathbb{R}).$$

Also, we denote by $\mathbb{Z}_+$ the set of all nonnegative integers, and, for every $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}_+^d$, we set

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d, \qquad \partial_x^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}, \qquad \partial_j := \frac{\partial}{\partial x_j},$$

and, for any natural number $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote by ∂_x^m any partial derivative of order m with respect to the space variables and set

$$\|\partial_x^m f\|_{L^p} := \sum_{|\alpha|=m} \|\partial_x^\alpha f\|_{L^p}.$$

Finally, for every $-\infty \leq a < b \leq \infty$, $\Omega \subset \mathbb{R}^d$, and any function $f : (a,b) \times \Omega \to \mathbb{R}$, $(t,x) \mapsto f(t,x)$, we denote by $f(t) : \Omega \to \mathbb{R}$, $t \in (a,b)$, the function $f(t)(x) = f(t,x)$, $x \in \Omega$.

With all these notations in mind, we can recall the concept of *mild solution* for the Cauchy problem (KS).

Definition 2.1. Given $u_0 \in L^1$ and $T \in (0,\infty)$, a function $u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}$ is said to be a mild solution of (KS) in $[0,T)$ if

(i) $u \in C([0,T); L^1) \cap C((0,T); L^{4/3})$,
(ii) $\sup_{0<t<T} \big(t^{1/4}\|u(t)\|_{4/3}\big) < \infty$,
(iii) u satisfies the integral equation

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds, \quad 0 < t < T, \tag{2.1}$$

where $e^{t\Delta}$ is the heat semigroup

$$(e^{t\Delta}f)(x) = \int_{\mathbb{R}^2} G(t, x-y)f(y)\,dy, \qquad G(t,x) = \frac{1}{4\pi t}\exp(-\frac{|x|^2}{4t}).$$

A function $u : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is said to be a global mild solution of (KS) with initial data u_0 if u is a mild solution of (KS) in $[0,T)$ for any $T \in (0,\infty)$.

The integral in (2.1) is well-defined by the well-known L^p-L^q estimates for the heat semigroup $e^{t\Delta}$ in $\mathbb{R}^2$

$$\|\partial_t^m \partial_x^n e^{t\Delta} f\|_p \le C t^{-1/q+1/p-m-n/2}\|f\|_q \qquad \text{for all} \ \ f \in L^q, \tag{2.2}$$

where $1 \le q \le p \le \infty$ and $m, n \in \mathbb{Z}_+$, and the fact that, for every $4/3 \le q < 2$,

$$\|f(\nabla N * g)\|_{2q/(4-q)} \le C_q \|f\|_q \|g\|_q \qquad \text{for all} \ \ f, g \in L^q, \tag{2.3}$$

where C_q is a positive constant depending only on q. Estimate (2.3) follows easily by combining the Hölder inequality with the next inequality of Hardy-Littlewood-Sobolev in $\mathbb{R}^2$

$$\|\frac{1}{|x|} * g\|_{2q/(2-q)} \le C_q \|g\|_q \qquad \text{for all} \ \ g \in L^q, \tag{2.4}$$

which is valid for all $1 < q < 2$, where $C_q > 0$ is a constant depending only on q. The integral in (2.1) is estimated as follows. By applying (2.2) with $m = 0$, $n = 1$, $q = 1$ and $p = 4/3$ and then (2.3) with $q = 4/3$,

$$\begin{aligned}
&\int_0^t \|\nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\|_{4/3}\,ds \\
&\le C\int_0^t (t-s)^{-1+3/4-1/2}\|u(s)(\nabla N * u)(s)\|_1\,ds \\
&\le C\int_0^t (t-s)^{-3/4}\|u(s)\|_{4/3}^2\,ds \\
&\le C\int_0^t (t-s)^{-3/4}s^{-1/2}\,ds \Big(\sup_{0<s<t} s^{1/4}\|u(s)\|_{4/3}\Big)^2 < \infty.
\end{aligned}$$

We remark that (KS) is rather similar to the Cauchy problem for the vorticity equation in $\mathbb{R}^2$

$$\begin{cases} \partial_t \omega = \Delta\omega - \nabla \cdot (\omega(\nabla^\perp N * \omega)), & t > 0,\ x \in \mathbb{R}^2, \\ \omega(0,x) = \omega_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{2.5}$$

where

$$\nabla^\perp N(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \quad x^\perp = (x_2, -x_1), \quad x = (x_1, x_2)$$

and $\nabla^\perp N * \omega$ is the velocity field of the Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$. For the global existence, uniqueness and regularity of solutions to the Cauchy problem (2.5), for example, see Giga-Miyakawa-Osada [22], Ben-Artzi [3], Kato [28]. There is a big difference between (2.5) and (KS) such as $\nabla\cdot(\nabla^\perp N * \omega) = 0$ in (2.5) and $\nabla\cdot(\nabla N * u) = -u$ in (KS), which makes differences in global solvability and large-time behavior. For example, given $\omega_0 \in L^1(\mathbb{R}^2)$ there exists a unique solution ω of (2.5) globally in time, which satisfies the following: For all $1 \le p \le \infty$,

$$\sup_{t>0} t^{1-1/p}\|\omega(t)\|_p < \infty,$$
$$\lim_{t\to\infty} t^{1-1/p}\|\omega(t) - MG(t)\|_p = 0,$$

where $M = \int_{\mathbb{R}^2} \omega_0\,dx$ and $G(t,x)$ is the heat kernel in $\mathbb{R}^2$. However, to get local existence, uniqueness and regularity for (KS), we rely on techniques used in [28] and [14], employing an approach that uses the heat semigroups $e^{t\Delta}$, combined with the estimate (2.3) of $f(\nabla N * g)$ involved in the nonlinear term of the equation in (KS).

To mention the existence, uniqueness and regularity of the mild solution of (KS), we must introduce some additional functional spaces going back to Kato [28]. For every $T > 0$, $1 \le p \le \infty$ and $\gamma \ge 0$, we will consider the Banach space $C_{\gamma,T}(L^p)$ with norm $\|\cdot\|_{p,\gamma,T}$, defined by

$$C_{\gamma,T}(L^p) := \{\, u \in C((0,T); L^p), \quad \sup_{0<t<T} (t^\gamma \|u(t)\|_p) < \infty \,\},$$
$$\|u\|_{p,\gamma,T} := \sup_{0<t<T} (t^\gamma \|u(t)\|_p) \quad \text{for all } u \in C_{\gamma,T}(L^p).$$

Also, for every $\gamma > 0$, we will consider the closed subspace of $C_{\gamma,T}(L^p)$ defined by

$$\dot{C}_{\gamma,T}(L^p) := \{\, u \in C_{\gamma,T}(L^p), \quad \lim_{t\to 0} (t^\gamma \|u(t)\|_p) = 0 \,\},$$

while, in case $\gamma = 0$, we take $\dot{C}_{0,T}(L^p) = BC([0,T); L^p)$.

The next result was established in [36, 38] by adapting the methods of [3, 14, 22, 28] for the vorticity equation in $\mathbb{R}^2$.

Proposition 2.1. *Suppose $u_0 \in L^1$. Then there exists $T = T(u_0) \in (0,\infty)$ such that the Cauchy problem* (KS) *has a unique mild solution u in $[0,T)$. Moreover, u satisfies the following properties:*

(i) *$u(t) \to u_0$ in L^1 as $t \to 0$.*
(ii) *For every $1 \le q \le \infty$, $u \in \dot{C}_{1-1/q,T}(L^q)$.*
(iii) *For every $\ell \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^2$ and $1 < q < \infty$, $\partial_t^\ell \partial_x^\alpha u \in \dot{C}_{1-1/q+|\alpha|/2+\ell,T}(L^q)$.*
(iv) *For every $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^2$ and $2 - \min\{1, |\alpha|\} < q < \infty$,*

$$\partial_t^\ell \partial_x^\alpha (\nabla N * u) \in \dot{C}_{1/2-1/q+|\alpha|/2+\ell,T}(L^q).$$

(v) *u is a classical solution of $\partial_t u = \Delta u - \nabla\cdot(u(\nabla N * u))$ in $(0,T)\times\mathbb{R}^2$.*
(vi) *$\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)dx$ for all $0 < t < T$.*
(vii) *If $u_0 > 0$ ($u_0 \ge 0$ but $u_0 \ne 0$), then $u(t,x) > 0$ for all $(t,x) \in (0,T)\times\mathbb{R}^2$.*

(viii) *If* $u_0 \log(1+|x|) \in L^1$, *then* $u(t)\log(1+|x|) \in L^1$ *for all* $0 < t < T$.

Remark 2.1. By Proposition 2.1, for every $u_0 \in L^1$ with $u_0 \log(1+|x|) \in L^1$, the Cauchy problem $(\mathrm{KS})_\psi$ admits a unique mild solution u, because, since $u(t)\log(1+|x|) \in L^1$, the function $\psi(t) := N * u(t)$ is well-defined in L^1_{loc} and it satisfies $-\Delta\psi = u$.

3. Decreasing rearrangements

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and $\theta \in \mathbb{R}$, we denote, by the sake of simplicity,

$$\{f > \theta\} := \{x \in \mathbb{R}^d : \ f(x) > \theta\}, \quad |f > \theta| := |\{x \in \mathbb{R}^d : \ f(x) > \theta\}|,$$

where $|A|$ stands for the Lebesgue measure of a measurable set A in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function vanishing at infinity in the sense that

$$||f| > \theta| < \infty \qquad \text{for all } \ \theta > 0.$$

Then, the *distribution function* μ_f of f is defined by

$$\mu_f(\theta) := ||f| > \theta|, \qquad \theta \geq 0,$$

the *decreasing rearrangement* f^* of f is defined through

$$f^*(s) := \inf\{\theta \geq 0 : \ \mu_f(\theta) \leq s\}, \qquad s \geq 0$$

(it is a generalized inverse of μ_f), and the *symmetric rearrangement*, or Schwarz symmetrization of f, denoted by $f^\sharp : \mathbb{R}^d \to \mathbb{R}$, is defined by

$$f^\sharp(x) := f^*(c_d|x|^d),$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. Some basic properties about rearrangements are the following:

(i) f^* is non-increasing and right-continuous on $[0,\infty)$.
(ii) $f^*(0) = \|f\|_{L^\infty(\mathbb{R}^d)}, \ \ f^*(\infty) = 0$.
(iii) $f^\sharp = f$ if f is radially symmetric and non-increasing in $|x|$.
(iv) f^* and $f^\sharp$ are globally continuous and bounded if f is globally continuous and bounded.
(v) $(f+g)^*(s_1+s_2) \leq f^*(s_1) + g^*(s_2)$ for all $s_1, s_2 > 0$.

The interested reader is sent to [2, 31, 34, 43] for these properties, and all subsequent ones on rearrangements. The next result collects some well-known properties.

Proposition 3.1. (i) *For every Borel measurable function* $\Phi : \mathbb{R} \to [0,\infty)$,

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\,dx = \int_{\mathbb{R}^d} \Phi(f^\sharp(x))\,dx = \int_0^\infty \Phi(f^*(s))\,ds.$$

(ii) *Let* $f, g : \mathbb{R}^d \to \mathbb{R}$ *be integrable on* $\mathbb{R}^d$ *such that*

$$\int_0^s f^*(\sigma)\,d\sigma \leq \int_0^s g^*(\sigma)\,d\sigma \qquad \text{for all } \ s > 0.$$

Then

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\,dx \leq \int_{\mathbb{R}^d} \Phi(|g(x)|)\,dx$$

for all convex functions $\Phi : [0,\infty) \to [0,\infty)$ *with* $\Phi(0) = 0$.

(iii) (The Hardy-Littlewood inequality) *Let* $p, q \in [1,\infty]$ *with* $1/p + 1/q = 1$. *Then, for every* $f \in L^p(\mathbb{R}^d)$ *and* $g \in L^q(\mathbb{R}^d)$,
$$\int_{\mathbb{R}^d} |f||g|\,dx \le \int_{\mathbb{R}^d} f^\sharp g^\sharp\,dx = \int_0^\infty f^* g^*\,ds.$$

(iv) (Contraction property) *For every* $p \in [1,\infty]$ *and* $f, g \in L^p(\mathbb{R}^d)$,
$$\|f^* - g^*\|_{L^p(0,\infty)} = \|f^\sharp - g^\sharp\|_{L^p(\mathbb{R}^d)} \le \|f - g\|_{L^p(\mathbb{R}^d)}.$$

(v) (The Pólya-Szegö inequality) *For every* $p \in [1,\infty]$ *and* $f \in W^{1,p}(\mathbb{R}^d)$, *one has that* $f^\sharp \in W^{1,p}(\mathbb{R}^d)$ *and*
$$\|\nabla f^\sharp\|_{L^p(\mathbb{R}^d)} \le \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Now, we collect Lemma 5.1 of [36].

Lemma 3.1. *Let* $v(t,x)$ *be a smooth function in* $(0,T) \times \mathbb{R}^2$, *radially symmetric in* x, *such that* $v(t) \in L^1 \cap L^\infty$ *for all* $t \in (0,T)$ *and*
$$\partial_t v = \Delta v - \nabla \cdot (v(\nabla N * v)) \quad \textit{in} \quad (0,T) \times \mathbb{R}^2.$$
Then the functions
$$\varphi(t,s) := v(t,x), \quad s = \pi|x|^2, \qquad \Phi(t,s) := \int_0^s \varphi(t,\sigma)\,d\sigma$$
satisfy
$$\int_{\mathbb{R}^2} v(t,x)\,dx = \int_0^\infty \varphi(t,s)ds, \qquad t \in [0,T), \tag{3.1}$$
$$\partial_t \varphi(t,s) = 4\pi \partial_s (s \partial_s \varphi(t,s)) + \partial_s \left(\varphi(t,s) \int_0^s \varphi(t,\sigma)\,d\sigma \right),$$
and
$$\partial_t \Phi(t,s) = 4\pi s \partial_s^2 \Phi(t,s) + \Phi(t,s) \partial_s \Phi(t,s). \tag{3.2}$$

The next two propositions collect some known properties of the decreasing rearrangement, with respect to the space variable x, of the solution of (KS) (see [19, 20, 36] for further details).

Proposition 3.2. *Let* u *be a nonnegative mild solution of* (KS) *in* $[0,T)$ *with nonnegative initial data* $u_0 \in L^1$, *let* u^* *denote its decreasing rearrangement with respect to* x, *and set*
$$H(t,s) := \int_0^s u^*(t,\sigma)\,d\sigma, \qquad 0 < t < T, \quad s \ge 0.$$
Then it hold that for every $p \in (1,\infty)$,

(i) $H(t,0) = 0$ *and* $H(t,\infty) = \int_{\mathbb{R}^2} u_0\,dx$ *for all* $0 < t < T$,

(ii) $H \in BC([0,T) \times [0,\infty))$ *and* $H(0,s) = \int_0^s u_0^*\,d\sigma$ *for all* $s > 0$,

(iii) $\partial_s H \in BC((T_0,T) \times (0,\infty)) \cap L^\infty(0,T; L^1(0,\infty))$ *for all* $0 < T_0 < T$,

(iv) $\partial_s^2 H \in L^\infty(T_0,T; L^p(s_0,\infty))$ *for all* $0 < T_0 < T$ *and* $s_0 > 0$,

(v) $\partial_t H \in L^\infty(T_0,T; L^p(0,R))$ *for all* $0 < T_0 < T$ *and* $R > 0$.

Proposition 3.3. *Let $H(t,s)$ be the one in Proposition 3.2. Then, for almost all $t \in (0,T)$,*

$$\partial_t H - 4\pi s \partial_s^2 H - H\partial_s H \leq 0 \qquad a.a.\ s > 0. \tag{3.3}$$

Proof. We give the outline of the proof.

We first remark that for almost all $0 < t < T$ and all $\theta > 0$,

$$\int_{\{u(t)>\theta\}} \partial_t u(t,x)\,dx = \int_0^{\mu(t,\theta)} \partial_t u^*(t,\sigma)\,d\sigma, \tag{3.4}$$

where $\mu(t,\theta) = |u(t) > \theta|$ (See Proposition 4.4, [36]). Take any $t \in (0,T)$ such that (3.4) holds, and let $\theta > 0$ be a regular value of $u(t)$, that is, $\nabla u(t,x) \neq 0$ for all $x \in \{x|\, u(t,x) = \theta\}$. Integrating equation $\partial_t u = \Delta u - \nabla\cdot(u(\nabla N * u))$ on $\{u(t) > \theta\}$ gives

$$\begin{aligned} -\int_{\{u(t)>\theta\}} \Delta u(t,x)\,dx = &-\int_{\{u(t)>\theta\}} \nabla\cdot(u(\nabla N * u))(t,x)\,dx \\ &-\int_{\{u(t)>\theta\}} \partial_t u(t,x)\,dx. \end{aligned} \tag{3.5}$$

By Green's formula,

$$\begin{aligned} &-\int_{\{u(t)>\theta\}} \nabla\cdot(u(\nabla N * u))(t,x)\,dx \\ &= \int_{\{u(t)=\theta\}} u(t,x)(\nabla N * u)(t,x)\cdot\frac{\nabla u(t,x)}{|\nabla u(t,x)|}\,d\mathcal{H}^1 \\ &= -\theta\int_{\{u(t)>\theta\}} \nabla\cdot(\nabla N * u)(t,x)\,dx = \theta\int_{\{u(t)>\theta\}} u(t,x)\,dx. \end{aligned}$$

Here we used $-\nabla\cdot(\nabla N * u) = u$. Noting that

$$\int_{\{u(t)>\theta\}} u(t,x)\,dx = \int_0^{\mu(t,\theta)} u^*(t,\sigma)\,d\sigma$$

and that $u^*(t,\mu(t,\theta)) = \theta$ by the continuity of the function $s \mapsto u^*(t,s)$ on $(0,\infty)$, we have

$$\begin{aligned} -\int_{\{u(t)>\theta\}} \nabla\cdot(u(\nabla N * u))(t,x)\,dx &= \theta\int_{\{u(t)>\theta\}} u(t,x)\,dx \\ &= u^*(t,\mu(t,\theta))\int_0^{\mu(t,\theta)} u^*(t,\sigma)\,d\sigma. \end{aligned} \tag{3.6}$$

By Green's formula,

$$\begin{aligned} -\int_{\{u(t)>\theta\}} \Delta u(t,x)\,dx &= \int_{\{u(t)=\theta\}} |\nabla u(t,x)|\,d\mathcal{H}^1 \\ &\geq 4\pi\mu(t,\theta)\frac{-1}{\partial_\theta\mu(t,\theta)}. \end{aligned} \tag{3.7}$$

Here we have used the following result:

- Let $f : \mathbb{R}^d \to [0,\infty)$ be of C^1, satisfying $f(x) \to 0$ as $|x| \to \infty$. If $\theta > 0$ is a regular value of f, that is, $|\nabla f| \neq 0$ on $f^{-1}(\theta)$, then the distribution function μ_f of f is differentiable at θ and satisfies
$$\mu_f'(\theta) = -\int_{\{f=\theta\}} \frac{1}{|\nabla f|}\, d\mathcal{H}^{d-1} \neq 0.$$
Moreover
$$\int_{\{f=\theta\}} |\nabla f|\, d\mathcal{H}^{d-1} \geq d^2 c_d^{2/d} \mu_f(\theta)^{2-2/d} \frac{-1}{\mu_f'(\theta)},$$
where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. For the proof, see [49] for example.

Substituting (3.4), (3.6) and (3.7) into (3.5) yields that
$$4\pi\mu(t,\theta)\frac{-1}{\partial_\theta\mu(t,\theta)} \leq u^*(t,\mu(t,\theta)) \int_0^{\mu(t,\theta)} u^*(t,\sigma)\, d\sigma - \int_0^{\mu(t,\theta)} \partial_t u^*(t,\sigma)\, d\sigma. \tag{3.8}$$
Hence, since the set of critical values θ of $u(t)$ is 1-dimensional measure zero by Sard's theorem, it follows from (3.8) that
$$4\pi \leq \mu(t,\theta)^{-1} F(t,\mu(t,\theta))\big(-\partial_\theta\mu(t,\theta)\big), \quad \text{a.a. } \theta \in (0, u^*(t,0)), \tag{3.9}$$
where
$$F(t,s) := u^*(t,s)\int_0^s u^*(t,\sigma)\, d\sigma - \int_0^s \partial_t u^*(t,\sigma)\, d\sigma.$$
The function $s \mapsto F(t,s)$ is continuous on $(0,\infty)$.

Let θ_1, θ_2 be any numbers such as $0 < \theta_1 < \theta_2 < u^*(t,0)$. Integrate (3.9) from θ_1 to θ_2 with respect to θ. Noting that $\theta \mapsto \mu(t,\theta)$ is non-increasing and $F(t,s) \geq 0$, we have
$$4\pi(\theta_2-\theta_1) \leq -\int_{\theta_1}^{\theta_2} \mu(t,\theta)^{-1} F(t,\mu(t,\theta))\partial_\theta\mu(t,\theta)\, d\theta \leq \int_{\mu(t,\theta_2)}^{\mu(t,\theta_1)} s^{-1}F(t,s)\, ds.$$
Let $0 < s' < s$ be such that $u^*(t,s) < u^*(t,s')$. For $0 < \varepsilon << 1$, we observe that
$$\mu(t,u^*(t,s)) \leq s, \quad \mu(t,u^*(t,s')-\varepsilon) \geq s'.$$
We take $\theta_1 = u^*(t,s), \theta_2 = u^*(t,s')-\varepsilon$ to get
$$4\pi(u^*(t,s')-\varepsilon-u^*(t,s)) \leq \int_{s'}^{s} \sigma^{-1}F(t,\sigma)\, d\sigma,$$
and by letting $\varepsilon \to 0$,
$$4\pi(u^*(t,s')-u^*(t,s)) \leq \int_{s'}^{s} \sigma^{-1}F(t,\sigma)\, d\sigma.$$
From this it follows that
$$-4\pi\partial_s u^*(t,s) \leq s^{-1}F(t,s), \quad \text{a.a. } s > 0,$$

and hence

$$-4\pi s\partial_s u^*(t,s) \le u^*(t,s)\int_0^s u^*(t,\sigma)\,d\sigma - \int_0^s \partial_t u^*(t,\sigma)\,d\sigma, \quad \text{a.a. } s>0.$$

This inequality implies (3.3) because $H(t,s) = \int_0^s u^*(t,\sigma)\,d\sigma$. □

By virtue of (3.3), we have the following comparison principle.

Proposition 3.4. *Suppose u is a nonnegative mild solution of* (KS) *in* $[0,T)$ *with nonnegative initial data $u_0 \in L^1$ and v is a nonnegative radially symmetric mild solution to* (KS) *with initial data $v_0 \in L^1$, $v_0 \ge 0$. Set*

$$v_0(x) := \varphi_0(\pi|x|^2), \qquad v(t,x) := \varphi(t,\pi|x|^2).$$

Then, the estimate

$$\int_0^s u_0^*(\sigma)\,d\sigma \le \int_0^s \varphi_0(\sigma)\,d\sigma \quad \textit{for all} \;\; s>0, \tag{3.10}$$

implies

$$\int_0^s u^*(t,\sigma)\,d\sigma \le \int_0^s \varphi(t,\sigma)\,d\sigma \quad \textit{for all} \;\; 0<t<T \;\textit{and}\; s>0.$$

4. Stationary solutions

We consider the stationary problem to (KS): $\varphi \in C^2(\mathbb{R}^2)$ and

$$\Delta\varphi - \nabla\cdot(\varphi(\nabla N * \varphi)) = 0 \quad \text{in } \mathbb{R}^2, \tag{4.1}$$

where

$$(\nabla N * \varphi)(x) := \int_{\mathbb{R}^2} \nabla N(x-y)\varphi(y)\,dy = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\varphi(y)\,dy.$$

We assume the following on φ:

$$\varphi(x) > 0 \;(x\in\mathbb{R}^2), \quad \varphi \in L^1\cap L^\infty. \tag{4.2}$$

Theorem 4.1. *Let $\varphi \in C^2(\mathbb{R}^2)$ be a solution of* (4.1) *and assume* (4.2). *Then φ is radially symmetric and has the following form: For $b>0, x_0\in\mathbb{R}^2$,*

$$\varphi(x) = \frac{8b}{(|x-x_0|^2+b)^2} \quad (x\in\mathbb{R}^2). \tag{4.3}$$

Hence $\int_{\mathbb{R}^2}\varphi\,dx = 8\pi$.

To prove this theorem, we begin with the following lemma whose proof follows from the argument by [16] (see also Lemma 2.1 in [39] for example).

Lemma 4.1. *For $f \in L^1\cap L^\infty$, define the function v on $\mathbb{R}^2$ by*

$$v(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} (\log|x-y| - \log|y|)f(y)\,dy. \tag{4.4}$$

Then

$$\lim_{|x|\to\infty} \frac{v(x)}{\log|x|} = \frac{M}{2\pi}, \tag{4.5}$$

where $M := \int_{\mathbb{R}^2} f\,dx$.

Proposition 4.1. *For φ in Theorem 4.1, define the function w on $\mathbb{R}^2$ by*

$$w(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}(\log|x-y| - \log|y|)\varphi(y)\,dy. \tag{4.6}$$

Then $\varphi = \lambda e^w$ on $\mathbb{R}^2$ for some $\lambda > 0$.

Proof. To prove this proposition, we claim

$$\int_{\mathbb{R}^2}\varphi|\nabla(\log\varphi - w)|^2\,dx = 0. \tag{4.7}$$

Once we get this claim, by $\varphi(x) > 0$ ($x \in \mathbb{R}^2$), we have

$$\nabla(\log\varphi - w) = 0 \quad \text{on} \ \ \mathbb{R}^2.$$

From this, $\log\varphi - w = Const.$ on $\mathbb{R}^2$, and hence, $\varphi = \lambda e^w$ on $\mathbb{R}^2$ for some $\lambda > 0$.

In what follows, we will prove this claim. Let ϕ be in $C_0^\infty([0,\infty))$ such that

$$\phi(r) = 1 \ (0 \le r \le 1), \quad 0 < \phi(r) < 1 \ (1 < r < 2), \quad \phi(r) = 0 \ (r \ge 2).$$

Observing $\phi^{1/6} \in C_0^1([0,\infty))$, we have

$$|\phi'(r)| = 6\phi^{5/6}(r)|(\phi^{1/6})'(r)| \le C\phi^{5/6}(r).$$

For $R > 1$, define the function $\phi_R \in C_0^\infty(\mathbb{R}^2)$ by

$$\phi_R(x) = \phi(|x|/R) \quad (x \in \mathbb{R}^2).$$

Then

$$\phi_R(x) = 1 \ (0 \le |x| \le R), \quad 0 < \phi_R(x) < 1 \ (R < |x| < 2R), \quad \phi_R(x) = 0 \ (|x| \ge 2R),$$

and there is a positive constant C independent of R such that

$$|\nabla\phi_R(x)| \le \frac{C}{R}\phi_R^{5/6}(x) \quad (x \in \mathbb{R}^2). \tag{4.8}$$

By the fact that

$$\nabla w(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\varphi(y)\,dy = (\nabla N * \varphi)(x),$$

we have

$$0 = \nabla\cdot(\nabla\varphi - \varphi(\nabla N * \varphi)) = \nabla\cdot(\varphi\nabla(\log\varphi - w)).$$

By this and the integration by parts, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2}\nabla\cdot(\varphi\nabla(\log\varphi - w))\,(\log\varphi - w)\phi_R\,dx \\ &= -\int_{\mathbb{R}^2}\varphi|\nabla(\log\varphi - w)|^2\phi_R\,dx - \int_{\mathbb{R}^2}\varphi(\log\varphi - w)\langle\nabla(\log\varphi - w), \nabla\phi_R\rangle\,dx, \end{aligned}$$

from which it follows that

$$\begin{aligned} \int_{\mathbb{R}^2}\varphi|\nabla(\log\varphi - w)|^2\phi_R\,dx &= -\int_{\mathbb{R}^2}\varphi(\log\varphi - w)\langle\nabla(\log\varphi - w), \nabla\phi_R\rangle\,dx \\ &= I. \end{aligned} \tag{4.9}$$

By (4.8) and Hölder's inequality, we estimate the right-hand side of (4.9) as follows.

$$
\begin{aligned}
|I| &\le \frac{C}{R}\int_{R<|x|<2R} \varphi|\log\varphi - w||\nabla(\log\varphi - w)|\phi_R^{5/6}\,dx \\
&\le \frac{C}{R}\Big(\int_{R<|x|<2R} \varphi|\log\varphi - w|^2\phi_R^{2/3}\,dx\Big)^{1/2} \\
&\qquad \times \Big(\int_{R<|x|<2R} \varphi|\nabla(\log\varphi - w)|^2\phi_R\,dx\Big)^{1/2} \\
&\le \frac{1}{2}\int_{\mathbb{R}^2} \varphi|\nabla(\log\varphi - w)|^2\phi_R\,dx + \frac{C}{2R^2}\int_{R<|x|<2R} \varphi|\log\varphi - w|^2\phi_R^{2/3}\,dx.
\end{aligned}
\tag{4.10}
$$

Hence, by (4.9), (4.10) and $0 \le \phi_R \le 1$,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \varphi|\nabla(\log\varphi - w)|^2\phi_R\,dx \le \frac{C}{R^2}\int_{R<|x|<2R} \varphi|\log\varphi - w|^2\,dx \\
&\le \frac{C}{R^2}\int_{R<|x|<2R} \varphi|\log\varphi|^2\,dx + \frac{C}{R^2}\int_{R<|x|<2R} \varphi|w|^2\,dx \\
&= II + III.
\end{aligned}
\tag{4.11}
$$

To estimate II, we divide II into two parts:

$$
II = \frac{C}{R^2}\Big(\int_{\Omega_1} + \int_{\Omega_2}\Big)\varphi|\log\varphi|^2\,dx = II_1 + II_2,
$$

where

$$
\begin{aligned}
\Omega_1 &:= \{x \in \mathbb{R}^2 |\, R < |x| < 2R,\ 0 < \varphi(x) \le 1\}, \\
\Omega_2 &:= \{x \in \mathbb{R}^2 |\, R < |x| < 2R,\ \varphi(x) > 1\}.
\end{aligned}
$$

We estimate II_1 as follows.

$$
II_1 \le \frac{C}{R^2}\Big(\int_{\Omega_1} \varphi\,dx\Big)^{1/2}\Big(\int_{\Omega_1} \varphi|\log\varphi|^4\,dx\Big)^{1/2} \le \frac{C}{R}\Big(\int_{\Omega_1} \varphi\,dx\Big)^{1/2}.
$$

Here we have used $0 \le \varphi|\log\varphi|^4 \le C$ $(0 < \varphi \le 1)$. To estimate II_2, observing

$$
0 \le \varphi|\log\varphi|^2 \le C\varphi^2 \quad (\varphi > 1),
$$

we have

$$
II_2 \le \frac{C}{R^2}\int_{\Omega_2} \varphi^2\,dx,
$$

Then

$$
II \le \frac{C}{R}\|\varphi\|_1^{1/2} + \frac{C}{R^2}\|\varphi\|_2^2. \tag{4.12}
$$

To estimate III, by Lemma 4.1 we observe that there exists a constant $C >$ such that

$$
|w(x)| \le C + C\log(1 + |x|) \quad \text{on} \ \ \mathbb{R}^2.
$$

By this estimate, we derive that for $R >> 1$,

$$\begin{aligned} III &\le \frac{C}{R^2}\int_{R<|x|<2R} \varphi|w|^2\,dx \le \frac{C}{R^2}\int_{R<|x|<2R} \varphi(\log(1+|x|))^2\,dx \\ &\le C\Big(\frac{\log(1+2R)}{R}\Big)^2\|\varphi\|_1. \end{aligned} \tag{4.13}$$

Plugging (4.12) and (4.13) into (4.11) yields that

$$\int_{\mathbb{R}^2}\varphi|\nabla(\log\varphi-w)|^2\phi_R\,dx \le \frac{C}{R}\|\varphi\|_1^{1/2}+\frac{C}{R^2}\|\varphi\|_2^2+C\Big(\frac{\log(1+2R)}{R}\Big)^2\|\varphi\|_1. \tag{4.14}$$

From this we first obtain $\varphi|\nabla(\log\varphi-w)|^2\in L^1$, and then, letting $R\to\infty$ in (4.14),

$$\int_{\mathbb{R}^2}\varphi|\nabla(\log\varphi-w)|^2\,dx=0.$$

Thus our claim (4.7) is established and the proof is complete. □

To prove Theorem 4.1, we need the following theorem by Chen-Li [15].

Theorem 4.2. *Let $v\in C^2(\mathbb{R}^2)$ be a solution of the following problem:*

$$\begin{cases} \Delta v+e^v=0, & x\in\mathbb{R}^2,\\ \displaystyle\int_{\mathbb{R}^2} e^v\,dx<\infty. \end{cases}$$

Then v is radially symmetric and has the following form:

$$v(x)=\log\frac{32\mu^2}{(\mu^2|x-y_0|^2+4)^2},\qquad \mu>0,\ y_0\in\mathbb{R}^2.$$

Proof of Theorem 4.1. For the function w defined by (4.6), by Proposition 4.1,

$$\varphi(x)=\lambda e^{w(x)}\ \text{ in } \mathbb{R}^2 \ \text{ for some }\ \lambda>0.$$

Since w belongs to $C^2(\mathbb{R}^2)$ and satisfies

$$\Delta w+\varphi=0 \quad\text{in}\ \ \mathbb{R}^2,$$

we have

$$\Delta w+\lambda e^w=0 \ \text{ in } \mathbb{R}^2. \tag{4.15}$$

Put $v(x)=w(\lambda^{-1/2}x)$ $(x\in\mathbb{R}^2)$. Then it follows from (4.15) and $\varphi=\lambda e^w\in L^1$ that

$$\Delta v+e^v=0\ \text{ in } \mathbb{R}^2,\qquad \int_{\mathbb{R}^2}e^v\,dx<\infty.$$

By Theorem 4.2, there exist $\mu>0$ and $y_0\in\mathbb{R}^2$ such that

$$v(x)=\log\frac{32\mu^2}{(\mu^2|x-y_0|^2+4)^2}\ \text{ in } \mathbb{R}^2.$$

By the definition of v,

$$w(x) = v(\lambda^{1/2}x) = \log \frac{32\mu^2}{(\mu^2|\lambda^{1/2}x - y_0|^2 + 4)^2}$$
$$= \log \frac{32\lambda^{-2}\mu^{-2}}{(|x - \lambda^{-1/2}y_0|^2 + 4\lambda^{-1}\mu^{-2})^2}.$$

Putting $b = 4\lambda^{-1}\mu^{-2} > 0$ and $x_0 = \lambda^{-1/2}y_0$, we have

$$w(x) = \log \frac{8b\lambda^{-1}}{(|x - x_0|^2 + b)^2},$$

and then

$$\varphi(x) = \lambda e^{w(x)} = \frac{8b}{(|x - x_0|^2 + b)^2}.$$

Thus we conclude (4.3). □

5. Dynamics of (KS) with critical mass 8π

In this case, by $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$, the second moment identity (1.4) implies that

$$\int_{\mathbb{R}^2} |x|^2 u(t,x)\,dx = \int_{\mathbb{R}^2} |x|^2 u_0(x)\,dx, \quad t > 0. \tag{5.1}$$

Under $|x|^2 u_0 \in L^1$, Blanche-Carrillo-Masmoudi [11] proved the following.

Theorem 5.1. *Let u_0 be in L^1 and nonnegative on $\mathbb{R}^2$ and $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$. Suppose that*

$$u_0 \log u_0,\ |x|^2 u_0 \in L^1.$$

Then there exists a nonnegative weak solution of $(\mathrm{KS})_\psi$ globally in time such that

$$\lim_{t\to\infty} u(t,x)dx = 8\pi\delta_{x_0}(x) \quad \text{in the sense of measure},$$

where δ_{x_0} is the Dirac distribution at x_0 and x_0 is the center of mass of u_0, namely

$$x_0 = \frac{1}{8\pi}\int_{\mathbb{R}^2} x u_0(x)\,dx.$$

We remark that in the context of mild solutions, Theorem 5.1 holds without $u_0 \log u_0 \in L^1$, namely $u_0, |x|^2 u_0 \in L^1$. In fact, the mild solution $u(t)$ belongs to L^p for $t > 0$ and for all $1 \le p \le \infty$ by Proposition 2.1, and hence, by (5.1) and Lemma 5.1 below, $u(t)\log u(t)$ belongs to L^1 for $t > 0$.

Lemma 5.1. *If a nonnegative function $f \in L^1$ satisfies*

$$f \log(1 + |x|), (1 + f)\log(1 + f) \in L^1,$$

then

$$\int_{\mathbb{R}^2} f|\log f|\,dx \le \int_{\mathbb{R}^2} (1+f)\log(1+f)\,dx + 2\alpha \int_{\mathbb{R}^2} f \log(2 + |x|)\,dx + \frac{1}{e}\int_{\mathbb{R}^2} \frac{1}{(2+|x|)^\alpha}\,dx, \tag{5.2}$$

where $2 < \alpha < \infty$.

Proof. We claim that for $a \geq 0, b > 0$,

$$a|\log a| \leq (1+a)\log(1+a) + 2a|\log b| + e^{-1}b. \tag{5.3}$$

In fact, since $|(a/b)\log(a/b)| \leq e^{-1}$ for $a/b \leq 1$, we have

$$a|\log a| \leq e^{-1}b + a|\log b|.$$

By $|\log(a/b)| \leq |\log((a+1)/b)|$ for $a/b > 1$,

$$|\log a| \leq \log(1+a) + 2|\log b|.$$

Hence

$$a|\log a| \leq (1+a)\log(1+a) + 2a|\log b|,$$

Thus we obtain (5.3).

Putting $a = f(x), b = (2+|x|)^{-\alpha}$ $(2 < \alpha < \infty)$ in (5.3) yields that

$$f(x)|\log f(x)| \leq (1+f(x))\log(1+f(x)) + 2\alpha f(x)\log(2+|x|) + e^{-1}(2+|x|)^{-\alpha}.$$

Integrating this inequality on $\mathbb{R}^2$, we conclude (5.2). □

We recall that the stationary solutions

$$w_{b,x_0}(x) = \frac{8b}{(|x-x_0|^2+b)^2} \quad (x \in \mathbb{R}^2)$$

satisfy

$$\int_{\mathbb{R}^2} |x| w_{b,x_0}(x)\,dx < \infty, \qquad \int_{\mathbb{R}^2} |x|^2 w_{b,x_0}(x)\,dx = \infty.$$

Taking into account Theorem 5.1, convergence to a stationary solution should be discussed for a class of nonnegative initial data $u_0 \in L^1$ satisfying $|x|^2 u_0 \notin L^1$. To study convergence to a stationary solution, Blanchet-Carlen-Carrillo [10] introduced the following Lyapunov functional $\mathcal{H}_{b,x_0}$ defined by

$$\mathcal{H}_{b,x_0}[f] = \int_{\mathbb{R}^2} \left(\sqrt{f(x)} - \sqrt{w_{b,x_0}(x)}\right)^2 w_{b,x_0}^{-1/2}(x)\,dx \quad \text{for } f \in L^1,\ f \geq 0, \tag{5.4}$$

which is the relative entropy of the fast diffusion equation

$$\partial_t u = \Delta\sqrt{u} + 2\sqrt{\frac{\pi}{bM}}\nabla\cdot(xu)$$

with respect to the stationary solution w_{b,x_0}. We remark that if $\mathcal{H}_{b,x_0}[f] < \infty$ for $f \in L^1,\ f \geq 0$, then

$$\int_{\mathbb{R}^2} |x| f(x)\,dx < \infty, \qquad \int_{\mathbb{R}^2} |x|^2 f(x)\,dx = \infty.$$

When x_0 is the origin, we denote w_{b,x_0} and $\mathcal{H}_{b,x_0}[f]$ by w_b and $\mathcal{H}_b[f]$, respectively, namely,

$$w_b(x) = \frac{8b}{(|x|^2+b)^2} \quad (x \in \mathbb{R}^2),$$

$$\mathcal{H}_b[f] = \int_{\mathbb{R}^2} \left(\sqrt{f(x)} - \sqrt{w_b(x)}\right)^2 w_b^{-1/2}(x)\,dx.$$

In this lecture, we will introduce and prove the following theorems by Löpez Gömez-Nagai-Yamada [27].

Theorem 5.2. *Let $u_0 \in L^1$ be a nonnegative initial data satisfying $\int_{\mathbb{R}^2} u_0\, dx = 8\pi$. Assume that $\mathcal{H}_b[u_0] < \infty$ for some $b > 0$. Then, the unique (nonnegative) mild solution u of* (KS) *is globally defined in time and for any $\tau > 0$ there exists $b_\tau > 0$ such that, for every $1 \le p \le \infty$,*

$$\|u(t)\|_p \le \|w_{b_\tau}\|_p \quad \text{for all } \; t \ge \tau. \tag{5.5}$$

If, in addition, $u_0 \in L^\infty$, then there also exists $b_0 > 0$ such that

$$\|u(t)\|_p \le \|w_{b_0}\|_p$$

for all $t \ge 0$ and $1 \le p \le \infty$.

Theorem 5.3. *Let $u_0 \in L^1$ be a nonnegative initial data satisfying $\int_{\mathbb{R}^2} u_0\, dx = 8\pi$, and assume that $\mathcal{H}_b[u_0] < \infty$ for some $b > 0$. Then the unique nonnegative mild solution u of* (KS) *converges to the stationary solution w_{b,x_0} in L^p for every $1 \le p \le \infty$ as $t \to \infty$, where x_0 is the center of mass of u_0.*

Such results as Theorems 5.2 and 5.3 were first proved by Blanchet-Carlen-Carrillo [10]. They assumed

$$F[u_0] := \int_{\mathbb{R}^2} u_0(x) \log u_0(x)\, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u_0(x) u_0(y) \log |x - y|\, dx dy < \infty,$$
$$\mathcal{H}_b[u_0] < \infty \quad \text{for some } \; b > 0,$$

and proved that

$$\sup_{t \ge \tau} \|u(t)\|_p < \infty \quad \text{for all } \; \tau > 0 \text{ and } \; 1 \le p < \infty$$

and u converges to w_{b,x_0} in L^1 as $t \to \infty$.

To prove their results in [10], they used, for constructing the solution of (KS), an involved discrete variational scheme, attributable to Jordan-Kinderlehrer-Otto [26], which make the proof extremely lengthily and fraught with technical difficulties. Our proofs in Löpez Gömez-Nagai-Yamada [27] rely on an appropriate treatment of the functional $\mathcal{H}_b$ through some classical rearrangement techniques and energy methods. So, our methods are radically different from those used by Blanchet-Carlen-Carrillo in [10]. However, as in [10], the entropy-entropy dissipation inequality

$$\mathcal{H}_b[u(t)] + \int_0^t \mathcal{D}[u(s)]\, ds \le \mathcal{H}_b[u_0] \qquad \text{for all } t > 0, \tag{5.6}$$

where $\mathcal{D}$ stands for the entropy dissipation functional

$$\mathcal{D}[u] := 8 \int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\, dx - \int_{\mathbb{R}^2} u^{3/2}\, dx,$$

will play an essential role in our analysis.

We summarize the dynamics of (KS) with critical mass known so far.

We put

$$
\begin{aligned}
&L^1_{+\mathrm{cri}} := \{f \in L^1 \,|\, f \geq 0 \text{ on } \mathbb{R}^2,\ \int_{\mathbb{R}^2} f\,dx = 8\pi\},\\
&\mathcal{M}_2 := \{f \in L^1_{+\mathrm{cri}} |\ \int_{\mathbb{R}^2} |x|^2 f(x)\,dx < \infty\},\\
&\mathcal{H}_{\mathrm{finite}} := \{f \in L^1_{+\mathrm{cri}} \,|\, \mathcal{H}_b[f] < +\infty \text{ for some } b > 0\},\\
&\mathcal{MH}_\infty := \{f \in L^1_{+\mathrm{cri}} \,|\, f \notin \mathcal{M}_2,\ \mathcal{H}_b[f] = +\infty \text{ for all } b > 0\}.
\end{aligned}
$$

Then

$$L^1_{+\mathrm{cri}} = \mathcal{M}_2 \cup \mathcal{H}_{\mathrm{finite}} \cup \mathcal{MH}_\infty.$$

(i) If $u_0 \in \mathcal{M}_2$, then u converges to $8\pi\delta_{x_0}$ as $t \to \infty$, where x_0 is the center of mass of u_0.
(ii) If $u_0 \in \mathcal{H}_{\mathrm{finite}}$, then u converges to a stationary solution w_{b,x_0} as $t \to \infty$.
(iii) There exists an initial data $u_0 \in \mathcal{MH}_\infty$ for which the omega limit set of u_0 with respect to L^∞-topology contains two different stationary solutions.

5.1. **Some properties of the entropy functional $\mathcal{H}_{b,x_0}$.** We begin by establishing some elementary properties of the entropy functional $\mathcal{H}_{b,x_0}$ defined by (5.4).

Lemma 5.2. *Suppose $b > 0$, $x_0 \in \mathbb{R}^2$ and $f \in L^1$ satisfies $f \geq 0$. Then,*
(i) *$\mathcal{H}_{b,x_0}[w_{b,x_0}] = 0$ and $\mathcal{H}_{b,x_0}[w_{a,x_0}] = \infty$ for all $a > 0$, $a \neq b$,*
(ii) *$\mathcal{H}_{b,x_0}[f] < \infty$ implies $\mathcal{H}_{b,x_1}[f] < \infty$ for all $x_1 \in \mathbb{R}^2$,*
(iii) *$\mathcal{H}_{b,x_0}[f] < \infty$ implies $\mathcal{H}_{a,x_0}[f] = \infty$ for all $a > 0$, $a \neq b$,*
(iv) *$\mathcal{H}_{b,x_0}[f] < \infty$ implies*

$$\int_{\mathbb{R}^2} \sqrt{b + |x|^2} f(x)\,dx \leq 16\pi b^{1/2} + (8b)^{1/4}\big(\|f\|_1^{1/2} + \|w_b\|_1^{1/2}\big)\sqrt{\mathcal{H}_b[f]}$$

and, in particular, $|x|f \in L^1$, though $|x|^2 f \notin L^1$.

Proof. For $a, b > 0$ with $a \neq b$ and sufficiently large $|x|$, there exists a constant $C > 0$ such that

$$\left(\sqrt{w_{a,x_0}(x)} - \sqrt{w_{b,x_0}(x)}\right)^2 w_{b,x_0}^{-1/2}(x) \geq \frac{C}{|x|^2}$$

and, hence, $\mathcal{H}_{b,x_0}[w_{a,x_0}] = \infty$. By definition, $\mathcal{H}_{b,x_0}[w_{b,x_0}] = 0$. This ends the proof of (i). Property (ii) follows easily from the fact that

$$\lim_{|x|\uparrow\infty} \frac{\left(\sqrt{f(x)} - \sqrt{w_{b,x_0}(x)}\right)^2 w_{b,x_0}^{-1/2}(x)}{\left(\sqrt{f(x)} - \sqrt{w_{b,x_1}(x)}\right)^2 w_{b,x_1}^{-1/2}(x)} = 1.$$

To prove (iii), let a, $b > 0$ with $a \neq b$. Then, it follows from

$$(z - x)^2 + (z - y)^2 \geq \frac{1}{2}(x - y)^2, \qquad x, y, z \in \mathbb{R},$$

that

$$\left(\sqrt{f} - \sqrt{w_{a,x_0}}\right)^2 w_{a,x_0}^{-1/2} \geq \frac{1}{2}\left(\sqrt{w_{b,x_0}} - \sqrt{w_{a,x_0}}\right)^2 w_{a,x_0}^{-1/2} - \left(\sqrt{f} - \sqrt{w_{b,x_0}}\right)^2 w_{a,x_0}^{-1/2}$$

in $\mathbb{R}^2$. Moreover, there exists a constant $C > 0$ such that

$$\left(\sqrt{f} - \sqrt{w_{b,x_0}}\right)^2 w_{a,x_0}^{-1/2} \leq C \left(\sqrt{f} - \sqrt{w_{b,x_0}}\right)^2 w_{b,x_0}^{-1/2}.$$

Therefore, integrating these estimates in $\mathbb{R}^2$, yields to

$$\mathcal{H}_{a,x_0}[f] \geq \frac{1}{2}\mathcal{H}_{a,x_0}[w_{b,x_0}] - C\mathcal{H}_{b,x_0}[f].$$

As, owing to (i), $\mathcal{H}_{a,x_0}[w_{b,x_0}] = \infty$, we find from this estimate that $\mathcal{H}_{a,x_0}[f] = \infty$, which concludes the proof of Part (iii).

Our proof of the estimate of Part (iv) is based on the proof of Lemma 1.10 of [10]. By the sake of completeness, we will give complete details here. Setting

$$I_1 := \int_{\mathbb{R}^2} \sqrt{b + |x|^2}\, w_b(x)\, dx, \qquad I_2 := \int_{\mathbb{R}^2} \sqrt{b + |x|^2}\, (f(x) - w_b(x))\, dx,$$

it is apparent that

$$\int_{\mathbb{R}^2} \sqrt{b + |x|^2} f(x)\, dx = I_1 + I_2.$$

By changing to polar coordinates, it is easily seen that $I_1 = 16\pi\sqrt{b}$. Moreover, as

$$\sqrt{b + |x|^2} = (8b)^{1/4} w_b^{-1/4}(x),$$

we have that

$$\begin{aligned}
|I_2| &\leq \int_{\mathbb{R}^2} \sqrt{b + |x|^2}\, |f(x) - w_b(x)|\, dx \\
&= (8b)^{1/4} \int_{\mathbb{R}^2} \left|\sqrt{f(x)} + \sqrt{w_b(x)}\right| \left|\sqrt{f(x)} - \sqrt{w_b(x)}\right| w_b^{-1/4}(x)\, dx \\
&\leq (8b)^{1/4} \left(\int_{\mathbb{R}^2} \left(\sqrt{f} + \sqrt{w_b}\right)^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^2} \left(\sqrt{f} - \sqrt{w_b}\right)^2 w_b^{-1/2}\, dx\right)^{1/2} \\
&= (8b)^{1/4} \|\sqrt{f} + \sqrt{w_b}\|_2 \sqrt{\mathcal{H}_b[f]} \leq (8b)^{1/4} \left(\|\sqrt{f}\|_2 + \|\sqrt{w_b}\|_2\right) \sqrt{\mathcal{H}_b[f]} \\
&= (8b)^{1/4} \left(\|f\|_1^{1/2} + \|w_b\|_1^{1/2}\right) \sqrt{\mathcal{H}_b[f]}.
\end{aligned}$$

Adding these estimates provides us with the estimate of Part (iv), which implies $|x|f \in L^1$. It remains to prove that $|x|^2 f \notin L^1$. Indeed, it follows from the definition of w_b that

$$\begin{aligned}
|x|^2 f(x) &= \sqrt{8b}\, w_b^{-1/2}(x) f(x) - b f(x) \\
&\geq \sqrt{8b}\, w_b^{-1/2}(x) \left[\frac{1}{2} w_b(x) - \left(\sqrt{f(x)} - \sqrt{w_b(x)}\right)^2\right] - b f(x) \\
&= \sqrt{2b}\, w_b^{1/2}(x) - \sqrt{8b} \left(\sqrt{f(x)} - \sqrt{w_b(x)}\right)^2 w_b^{-1/2}(x) - b f(x).
\end{aligned}$$

Consequently, integrating in $\mathbb{R}^2$ shows that

$$\int_{\mathbb{R}^2} |x|^2 f(x)\, dx \geq \sqrt{2b} \int_{\mathbb{R}^2} \sqrt{w_b}\, dx - \sqrt{8b}\, \mathcal{H}_b[f] - b \int_{\mathbb{R}^2} f\, dx.$$

Therefore, $\int_{\mathbb{R}^2} |x|^2 f(x)\, dx = \infty$, because

$$\int_{\mathbb{R}^2} \sqrt{w_b}\, dx = \infty, \quad \mathcal{H}_b[f] < \infty, \quad \int_{\mathbb{R}^2} f\, dx < \infty.$$

The proof is complete. □

Lemma 5.3. *Suppose $f \in L^1$, $f \geq 0$, $\int_{\mathbb{R}^2} f = 8\pi$ and $\nabla f^{1/4} \in L^2$. Then, the entropy dissipation of f must be non-negative, i.e.,*

$$\mathcal{D}[f] := 8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2\, dx - \int_{\mathbb{R}^2} f^{3/2}\, dx \geq 0.$$

Moreover, $\mathcal{D}[f] = 0$ if and only if $f = w_{b,x_0}$ for some $b > 0$ and $x_0 \in \mathbb{R}^2$.

Lemma 5.3 follows readily by applying the next Gagliardo-Nirenberg-Sobolev estimate to the function $g := f^{1/4}$.

Lemma 5.4. *Suppose $g \in L^4$ and $|\nabla g| \in L^2$. Then,*

$$\pi \int_{\mathbb{R}^2} |g|^6\, dx \leq \int_{\mathbb{R}^2} |\nabla g|^2\, dx \int_{\mathbb{R}^2} |g|^4\, dx.$$

Moreover, the equality occurs if and only if $g = w_{b,x_0}^{1/4}$ for some $b > 0$ and $x_0 \in \mathbb{R}^2$.

Lemma 5.4 is a special case of the optimal Gagliardo-Nirenberg-Sobolev estimates established by Del Pino-Dolbeault in Theorem 1 of [18], by making the special choice $d = 2$ and $p = 3$ therein.

The next result establishes the entropy-entropy dissipation inequality (5.6) for the nonnegative mild solutions of (KS).

Theorem 5.4. *Let u_0 be such that*

$$u_0 \geq 0 \text{ on } \mathbb{R}^2, \qquad u_0 \in L^1, \qquad \int_{\mathbb{R}^2} u_0 = 8\pi, \tag{5.7}$$

and $\mathcal{H}_b[u_0] < \infty$ for some $b > 0$. Then the mild solution u of (KS) *in $[0, T)$ satisfies*

$$\mathcal{H}_b[u(t)] + \int_0^t \mathcal{D}[u(s)]\, ds \leq \mathcal{H}_b[u_0] \quad \text{for all } \ 0 < t < T, \tag{5.8}$$

where $\mathcal{D}[u]$ is defined by

$$\mathcal{D}[u] := 8 \int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\, dx - \int_{\mathbb{R}^2} u^{3/2}\, dx. \tag{5.9}$$

In what follows, we only give a formal proof of Theorem 5.4. See [10, 27] for a rigorous proof.

$$\frac{d}{dt}\mathcal{H}_b[u(t)] = \frac{d}{dt} \int_{\mathbb{R}^2} (\sqrt{u} - \sqrt{w_b})^2 w_b^{-1/2}\, dx = \int_{\mathbb{R}^2} \partial_t u (w_b^{-1/2} - u^{-1/2})\, dx.$$

By $w_b^{-1/2}(x) = (8b)^{-1/2}(|x|^2 + b)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_t u(t) w_b^{-1/2}\,dx &= (8b)^{-1/2} \int_{\mathbb{R}^2} \Delta u(t)(|x|^2 + b)\,dx \\
&\quad - (8b)^{-1/2} \int_{\mathbb{R}^2} \nabla \cdot (u(t)(\nabla N * u)(t))(|x|^2 + b)\,dx \\
&= (8b)^{-1/2} \int_{\mathbb{R}^2} u(t)\Delta|x|^2\,dx + 2(8b)^{-1/2} \int_{\mathbb{R}^2} u(t)x \cdot (\nabla N * u)(t)\,dx \\
&= 4(8b)^{-1/2} \int_{\mathbb{R}^2} u(t)\,dx \\
&\quad - 2(8b)^{-1/2} \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y) \frac{x \cdot (x-y)}{|x-y|^2}\,dydx.
\end{aligned}$$

By the symmetry of the function $u(t,x)u(t,y)x \cdot (x-y)/|x-y|^2$ with respect to x and y, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y) \frac{x \cdot (x-y)}{|x-y|^2}\,dydx \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y) \Big(\frac{x \cdot (x-y)}{|x-y|^2} + \frac{y \cdot (y-x)}{|x-y|^2} \Big)\,dydx \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y)\,dydx \\
&= \frac{1}{2} \Big(\int_{\mathbb{R}^2} u(t,x)\,dx \Big)^2.
\end{aligned}$$

Hence, since $\int_{\mathbb{R}^2} u(t)\,dx = 8\pi$, we have

$$\int_{\mathbb{R}^2} \partial_t u(t) w_b^{-1/2}\,dx = 4(8b)^{-1/2} \int_{\mathbb{R}^2} u(t)\,dx - (8b)^{-1/2} \frac{1}{2\pi} \Big(\int_{\mathbb{R}^2} u(t,x)\,dx \Big)^2 = 0.$$

Next,

$$\int_{\mathbb{R}^2} \partial_t u u^{-1/2}\,dx = \int_{\mathbb{R}^2} \Delta u u^{-1/2}\,dx - \int_{\mathbb{R}^2} \nabla \cdot (u(\nabla N * u)) u^{-1/2}\,dx.$$

Then

$$\int_{\mathbb{R}^2} \Delta u u^{-1/2}\,dx = \frac{1}{2} \int_{\mathbb{R}^2} u^{-3/2} |\nabla u|^2\,dx = 8 \int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\,dx$$

and

$$\begin{aligned}
&- \int_{\mathbb{R}^2} \nabla \cdot (u(\nabla N * u)) u^{-1/2}\,dx = -\frac{1}{2} \int_{\mathbb{R}^2} u^{-1/2} \nabla u \cdot (\nabla N * u)\,dx \\
&= - \int_{\mathbb{R}^2} \nabla u^{1/2} \cdot (\nabla N * u)\,dx = \int_{\mathbb{R}^2} u^{1/2} \nabla \cdot (\nabla N * u)\,dx = - \int_{\mathbb{R}^2} u^{3/2}\,dx.
\end{aligned}$$

Here we used $\nabla \cdot (\nabla N * u) = -u$. Hence

$$\int_{\mathbb{R}^2} \partial_t u u^{-1/2}\,dx = 8 \int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\,dx - \int_{\mathbb{R}^2} u^{3/2}\,dx = \mathcal{D}[u(t)].$$

Therefore

$$\frac{d}{dt}\mathcal{H}_b[u(t)] = -\mathcal{D}[u(t)],$$

from which (5.8) follows.

5.2. **Boundedness of the solutions.** As $f^\sharp = f$ if f is radially symmetric and non-increasing in $|x|$, we observe that

$$w_b(x) = w_b^\sharp(x) = w_b^*(\pi|x|^2), \quad x \in \mathbb{R}^2.$$

Here

$$w_b(x) = \frac{8b}{(|x|^2+b)^2}, \qquad x \in \mathbb{R}^2$$

is the stationary solution of (KS), and, therefore, the decreasing rearrangement of $w_b(x)$ is given by

$$w_b^*(s) = \frac{8\pi^2 b}{(s+\pi b)^2}, \qquad s \geq 0. \tag{5.10}$$

Consequently,

$$\int_0^s w_b^*\, d\sigma = \frac{8\pi s}{s+\pi b}, \qquad s \geq 0. \tag{5.11}$$

Naturally, this implies $\int_0^\infty w_b^*\, d\sigma = 8\pi$ and

$$\int_s^\infty w_b^*\, d\sigma = 8\pi - \frac{8\pi s}{s+\pi b} = \frac{8\pi^2 b}{s+\pi b}$$

and hence,

$$\liminf_{s\to\infty}\left(s\int_s^\infty w_b^*\, d\sigma\right) = 8\pi^2 b.$$

The next result establishes that, among all the nonnegative functions f with $\int_{\mathbb{R}^2} f\, dx = 8\pi$ satisfying

$$\liminf_{s\to\infty}\left(s\int_s^\infty f^*(\sigma)\, d\sigma\right) > 0, \tag{5.12}$$

the stationary profiles provide us with the f maximizing the mass function $\int_0^s f^*\, d\sigma$.

Lemma 5.5. *Suppose f satisfies*

$$f \geq 0 \quad in\ \mathbb{R}^2, \qquad f \in L^1, \qquad \int_{\mathbb{R}^2} f\, dx = 8\pi, \tag{5.13}$$

and (5.12). *Then there exist $b_0 > 0$ and $s_0 > 0$ such that*

$$\int_0^s f^*\, d\sigma < \int_0^s w_{b_0}^*\, d\sigma \quad \textit{for all} \;\; s \geq s_0.$$

If, in addition, $f \in L^\infty$, then there exists $b_1 \in (0, b_0)$ such that

$$\int_0^s f^*\, d\sigma < \int_0^s w_{b_1}^*\, d\sigma \quad \textit{for all} \;\; s > 0.$$

Proof. According to (5.12), there exist $b_0 > 0$ and $s_0 > 0$ such that

$$s \int_s^\infty f^* \, d\sigma > 8\pi^2 b_0 \quad \text{for all} \quad s \geq s_0,$$

which implies

$$\int_s^\infty f^* \, d\sigma > \frac{8\pi^2 b_0}{s + \pi b_0} \quad \text{for all} \quad s \geq s_0.$$

On the other hand, owing to Proposition 3.1, it follows from (5.13) that

$$\int_0^\infty f^* \, d\sigma = \int_{\mathbb{R}^2} f \, dx = 8\pi.$$

Thus, using (5.11), it becomes apparent that

$$\int_0^s f^* \, d\sigma = 8\pi - \int_s^\infty f^* \, d\sigma < 8\pi - \frac{8\pi^2 b_0}{s + \pi b_0} = \frac{8\pi s}{s + \pi b_0} = \int_0^s w_{b_0}^* \, d\sigma, \qquad s \geq s_0.$$

Subsequently, besides (5.12) and (5.13), we assume that $f \in L^\infty$. Naturally, for every $b_1 \in (0, b_0)$, we also have that

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_0}^* \, d\sigma = \frac{8\pi s}{s + \pi b_0} < \frac{8\pi s}{s + \pi b_1} = \int_0^s w_{b_1}^* \, d\sigma \qquad \text{for all} \quad s \geq s_0.$$

Let $b_1 < b_0$ be such that

$$0 < f^*(0) = \|f\|_{L^\infty(\mathbb{R}^2)} < 8/b_1.$$

Then there exists $\delta > 0$ such that

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_1}^* \, d\sigma = \frac{8\pi s}{s + \pi b_1} \qquad \text{for all} \quad s \in [0, \delta].$$

This completes the proof if $\delta \geq s_0$, but, in general, $\delta < s_0$. So, suppose $\delta < s_0$. We should shorten b_1, if necessary, so that

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_1}^* \, d\sigma = \frac{8\pi s}{s + \pi b_1} \qquad \text{for all} \quad s \in [\delta, s_0]. \tag{5.14}$$

Thanks to (5.12),

$$\int_0^s f^* \, d\sigma < \int_0^\infty f^* \, d\sigma = 8\pi \quad \text{for all} \quad s > 0.$$

On the other hand, we have that

$$\lim_{b_1 \downarrow 0} \frac{8\pi s}{s + \pi b_1} = 8\pi \qquad \text{uniformly in} \quad [\delta, s_0].$$

Consequently, b_1 can be shortened, if necessary, to get (5.14). This ends the proof. □

Theorem 5.5. *Let $u_0 \in L^1 \cap L^\infty$ be such that $u_0 \geq 0$, $\int_{\mathbb{R}^2} u_0 \, dx = 8\pi$, and*

$$\liminf_{s \to \infty} \left(s \int_s^\infty u_0^* \, d\sigma \right) > 0. \tag{5.15}$$

Then the (unique) nonnegative mild solution u of (KS) *is globally defined in time, and there exists $b > 0$ such that, for every $t > 0$, $s > 0$, and $p \in [1, \infty]$,*

$$\int_0^s u^*(\sigma, t)\, d\sigma \leq \int_0^s w_b^*(\sigma)\, d\sigma \quad \text{and} \quad \|u(t)\|_p \leq \|w_b\|_p. \tag{5.16}$$

Proof. According to Lemma 5.5, there exists $b > 0$ such that

$$\int_0^s u_0^*\, d\sigma < \int_0^s w_b^*\, d\sigma \quad \text{for all} \quad s > 0.$$

Thus, the first estimate of (5.16) follows from Proposition 3.4. Naturally, the second estimate follows from Proposition 3.1 (ii) by choosing an appropriate Φ. In particular, this shows the global existence of unique mild solution u. □

Naturally, Theorem 5.2 is a direct consequence from the following lemma.

Lemma 5.6. *Suppose f satisfies* (5.13) *and $\mathcal{H}_b[f] < \infty$ for some $b > 0$. Then,*

$$\liminf_{s \to \infty} \left(s \int_s^\infty f^*\, d\sigma \right) \geq 2\pi^2 b. \tag{5.17}$$

In particular, (5.12) *is satisfied.*

Proof. Setting

$$g := \sqrt{f} - \sqrt{w_b},$$

it is apparent that

$$f = w_b + h, \quad h := 2g\sqrt{w_b} + g^2. \tag{5.18}$$

Moreover,

$$\int_{\mathbb{R}^2} g^2(x)(b + |x|^2)\, dx = \sqrt{8b} \int_{\mathbb{R}^2} g^2(x) w_b^{-1/2}(x)\, dx = \sqrt{8b}\, \mathcal{H}_b[f] < \infty. \tag{5.19}$$

For every $R > 1$, we have that

$$\int_{|x| \geq R} g^2(x)\, dx \leq R^{-2} \int_{|x| \geq R} |x|^2 g^2(x)\, dx,$$

and, hence, by (5.19),

$$\int_{|x| \geq R} g^2(x)\, dx = o\Big(R^{-2}\Big) \qquad \text{as} \quad R \to \infty.$$

Similarly, since

$$\int_{|x| \geq R} \frac{w_b(x)}{|x|^2}\, dx = \int_{|x| \geq R} \frac{8b}{(b + |x|^2)^2 |x|^2}\, dx \leq 8b \int_{|x| \geq R} |x|^{-6}\, dx = 4\pi b R^{-4},$$

it follows from Hölder's inequality that

$$\begin{aligned}\int_{|x|\geq R}\sqrt{w_b(x)}|g(x)|\,dx &= \int_{|x|\geq R}\frac{\sqrt{w_b(x)}}{|x|}|g(x)||x|\,dx\\ &\leq \Big(\int_{|x|\geq R}\frac{w_b(x)}{|x|^2}\,dx\Big)^{1/2}\Big(\int_{|x|\geq R}|g(x)|^2|x|^2\,dx\Big)^{1/2}\\ &\leq 2\sqrt{\pi b}R^{-2}\Big(\int_{|x|\geq R}|g(x)|^2|x|^2\,dx\Big)^{1/2}\end{aligned}$$

and, consequently, (5.19) implies

$$\int_{|x|\geq R}\sqrt{w_b(x)}|g(x)|\,dx = o\Big(R^{-2}\Big)\quad \text{as }\ R\to\infty.$$

Therefore, we find from (5.18) that

$$\int_{|x|\geq R}|h(x)|\,dx = o\Big(R^{-2}\Big)\quad \text{as }\ R\to\infty. \tag{5.20}$$

As $w_b = f + (-h)$ and $(-h)^* = h^*$, applying property (iv) of the basic properties on rearrangements in Section 3, it is apparent that

$$w_b^*(2s) \leq f^*(s) + h^*(s) \qquad \text{for all }\ s > 0$$

and hence,

$$f^*(s) \geq w_b^*(2s) - h^*(s) \quad \text{for all }\ s > 0. \tag{5.21}$$

We will derive (5.17) from (5.21). To do it, we need to estimate $\int_s^\infty w_b^*(2\sigma)\,d\sigma$ and $\int_s^\infty h^*(\sigma)\,d\sigma$. By (5.10), we find that

$$\int_s^\infty w_b^*(2\sigma)\,d\sigma = \frac{4\pi^2 b}{2s+\pi b}$$

and, hence,

$$\lim_{s\to\infty}\left(s\int_s^\infty w_b^*(2\sigma)\,d\sigma\right) = 2\pi^2 b. \tag{5.22}$$

To conclude the proof of the lemma, it suffices to show that

$$\int_s^\infty h^*(\sigma)\,d\sigma \leq \int_{|x|\geq (s/\pi)^{1/2}}|h(x)|\,dx. \tag{5.23}$$

Indeed, suppose (5.23) holds. Then, by (5.20) we deduce that

$$s\int_s^\infty h^*(\sigma)\,d\sigma \leq s\int_{|x|\geq (s/\pi)^{1/2}}|h(x)|\,dx \to 0 \quad \text{as }\ s\to\infty \tag{5.24}$$

and, therefore, combining (5.21), (5.22) and (5.24), (5.17) holds.

The proof of (5.23) can be accomplished as follows. Thanks to the Hardy-Littlewood inequality, for every $R > 0$, we have that

$$\int_{|x|<R}|h(x)|\,dx = \int_{\mathbb{R}^2}|h(x)|\chi_{B_R}(x)\,dx \leq \int_{\mathbb{R}^2}h^\sharp(x)\chi_{B_R}^\sharp(x)\,dx = \int_{|x|<R}h^\sharp(x)\,dx,$$

where χ_{B_R} stands for the characteristic function of the ball $B_R := B_R(0)$, and we have used that $\chi^\sharp_{B_R} = \chi_{B_R}$. As, due to Proposition 3.1(i),

$$\int_{\mathbb{R}^2} |h|\,dx = \int_{\mathbb{R}^2} h^\sharp\,dx,$$

we infer from the previous estimate that

$$\int_{|x|\ge R} |h(x)|\,dx \ge \int_{\mathbb{R}^2} h^\sharp(x)\,dx - \int_{|x|<R} h^\sharp(x)\,dx = \int_{|x|\ge R} h^\sharp(x)\,dx.$$

Therefore, by the definition of $h^\sharp$,

$$\begin{aligned}\int_{|x|\ge R} |h(x)|\,dx \ge \int_{|x|\ge R} h^\sharp(x)\,dx &= \int_{|x|\ge R} h^*(\pi|x|^2)\,dx \\ &= 2\pi\int_R^\infty h^*(\pi\rho^2)\rho\,d\rho = \int_{\pi R^2}^\infty h^*(\sigma)\,d\sigma.\end{aligned}$$

Taking $s = \pi R^2$ in this inequality shows (5.23). □

Proof of Theorem 5.2. Let $T_{max} > 0$ denote the maximal existence time of the unique mild solution of (KS). By Proposition 2.1, $u(t) \in L^1 \cap L^\infty$ for all $t \in (0, T_{max})$. Moreover, by Lemma 5.3, we have that $\mathcal{D}(u(t)) \ge 0$ for all $t \in (0, T_{max})$. Thus, owing to Theorem 5.4, we have that

$$\mathcal{H}_b[u(t)] \le \mathcal{H}_b[u_0] < \infty \qquad \text{for all} \quad t \in (0, T_{max}). \tag{5.25}$$

Consequently, it follows from Lemma 5.6 that

$$\liminf_{s\to\infty} \left(s\int_s^\infty u^*(\tau,\sigma)\,d\sigma \right) \ge 2\pi^2 b.$$

As the function $t \mapsto u(t+\tau)$ is a mild solution of (KS) in $[0, T_{max}-\tau)$ with nonnegative initial data $u(\tau) \in L^1 \cap L^\infty$, according to Theorem 5.5 $u(t+\tau)$ must be globally defined in time and (5.5) holds. In particular, $T_{max} = \infty$ and the proof is complete. □

5.3. **Estimating the derivatives of the solutions.** Throughout this section, we consider a nonnegative initial data $u_0 \in L^1$ satisfying

$$\int_{\mathbb{R}^2} u_0\,dx = 8\pi \quad \text{and} \quad \mathcal{H}_b[u_0] < \infty \ \text{ for some } \ b > 0,$$

and denote by u the nonnegative global mild solution of (KS). By Theorem 5.2, we already know that

$$\sup_{t\ge 1} \|u(t)\|_p < \infty \quad \text{for all} \ \ 1 \le p \le \infty. \tag{5.26}$$

Lemma 5.7. *The following mass estimate for the gradient holds:*

$$\sup_{t\ge 1} \int_t^{t+1} \|\nabla u(s)\|_2^2\,ds < \infty. \tag{5.27}$$

Proof. Multiplying the differential equation of (KS) by u and integrating by parts on $\mathbb{R}^2$ shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} u^2\,dx &= \int_{\mathbb{R}^2} u\partial_t u\,dx = \int_{\mathbb{R}^2} u\Delta u\,dx - \int_{\mathbb{R}^2} u\nabla\cdot(u(\nabla N * u))\,dx \\ &= -\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla u^2, \nabla N * u\rangle\,dx \\ &= -\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^2} u^2\,\nabla\cdot(\nabla N * u)\,dx \\ &= -\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} u^3\,dx,\end{aligned}$$

where we have used $\nabla\cdot(\nabla N * u) = -u$. Integrating the previous identity on $[t, t+1]$ with respect to time, we find that

$$\int_t^{t+1}\|\nabla u(s)\|_2^2\,ds \le \frac{1}{2}\|u(t)\|_2^2 + \frac{1}{2}\int_t^{t+1}\|u(s)\|_3^3\,ds,$$

and, hence, (5.27) holds from (5.26). □

To proceed further, we need the following two lemmas.

Lemma 5.8. *For $a \in \mathbb{R}$, let f, g and h be nonnegative functions on $[a,\infty)$ such that*

$$\sup_{t\ge a}\int_t^{t+1}(f+h)\,ds < \infty. \tag{5.28}$$

If

$$f'(t) + g(t) \le bf(t) + h(t), \qquad t > a \tag{5.29}$$

for some positive constant b, where $f' = df/dt$, then

$$\sup_{t\ge a+1} f(t) < \infty \quad \textit{and} \quad \sup_{t\ge a+1}\int_t^{t+1} g(s)\,ds < \infty. \tag{5.30}$$

Proof. Differentiating with respect to time, we find that, for every $\tau \ge a$,

$$\frac{d}{dt}\big((t-\tau)f(t)\big) = (t-\tau)f'(t) + f(t) \le b(t-\tau)f(t) + f(t) + (t-\tau)h(t).$$

Thus, integrating this inequality on $[\tau, \tau+1]$ yields to

$$\begin{aligned}f(\tau+1) &\le b\int_\tau^{\tau+1}(t-\tau)f(t)\,dt + \int_\tau^{\tau+1} f(t)\,dt + \int_\tau^{\tau+1}(t-\tau)h(t)\,dt \\ &\le (b+1)\int_\tau^{\tau+1} f(t)\,dt + \int_\tau^{\tau+1} h(t)\,dt \le (b+1)\int_\tau^{\tau+1}(f+h)\,dt.\end{aligned} \tag{5.31}$$

Consequently, the first estimate of (5.30) follows from (5.28) and (5.31).

Next, for any $\tau \ge a+1$, integrating (5.29) on $[\tau, \tau+1]$ and applying

$$f(\tau) \le (b+1)\int_{\tau-1}^{\tau} f\,dt + \int_{\tau-1}^{\tau} h\,dt,$$

which is (5.31) with τ replaced by $\tau-1$, we have that

$$
\begin{aligned}
f(\tau+1)+\int_\tau^{\tau+1} g\,dt &\le f(\tau)+b\int_\tau^{\tau+1} f\,dt+\int_\tau^{\tau+1} h\,dt \\
&\le (b+1)\int_{\tau-1}^{\tau} f\,dt+\int_{\tau-1}^{\tau} h\,dt+b\int_\tau^{\tau+1} f\,dt \\
&\quad+\int_\tau^{\tau+1} h\,dt \\
&\le (b+1)\int_{\tau-1}^{\tau+1} f\,dt+\int_{\tau-1}^{\tau+1} h\,dt.
\end{aligned}
\tag{5.32}
$$

Therefore, the second estimate of (5.30) follows from (5.28) and (5.32). □

Lemma 5.9. *For all $f\in L^1\cap L^\infty$, $\nabla N * f$ belongs to L^∞. Precisely,*

$$\|\nabla N * f\|_\infty \le (2/\pi)^{1/2}\|f\|_1^{1/2}\|f\|_\infty^{1/2}. \tag{5.33}$$

Proof. For every $A>0$,

$$
\begin{aligned}
2\pi|(\nabla N * f)(x)| &\le \Big(\int_{|x-y|\le A}+\int_{|x-y|>A}\Big)\frac{|f(y)|}{|x-y|}\,dy \\
&\le 2\pi\|f\|_\infty A+\|f\|_1A^{-1}.
\end{aligned}
$$

From this we have

$$2\pi|(\nabla N * f)(x)| \le 2(2\pi\|f\|_\infty\|f\|_1)^{1/2},$$

which implies (5.33). □

Proposition 5.1. *The following estimates hold:*

$$\sup_{t\ge 2}\Big(\|\nabla u(t)\|_2+\int_t^{t+1}\|\Delta u(s)\|_2^2\,ds\Big)<\infty, \tag{5.34}$$

$$\sup_{t\ge 2}\int_t^{t+1}\|\partial_t u(s)\|_2^2\,ds<\infty. \tag{5.35}$$

Proof. Integrating by parts, we have that

$$
\begin{aligned}
\frac12\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla u(t)|^2\,dx &= -\int_{\mathbb{R}^2}\Delta u(\partial_t u)\,dx \\
&= -\int_{\mathbb{R}^2}|\Delta u|^2\,dx+\int_{\mathbb{R}^2}\Delta u\nabla\cdot(u(\nabla N * u))\,dx \\
&= -\int_{\mathbb{R}^2}|\Delta u|^2\,dx+\int_{\mathbb{R}^2}\Delta u\langle\nabla u,\nabla N * u\rangle\,dx \\
&\quad-\int_{\mathbb{R}^2}\Delta u\,u^2\,dx,
\end{aligned}
$$

where we have invoked to $\nabla \cdot (\nabla N * u) = -u$. Thus, by Hölder's inequality and Young's inequality, we find that

$$\begin{aligned}\int_{\mathbb{R}^2} |\Delta u||\langle \nabla u, \nabla N * u\rangle|\,dx &\le \Big(\int_{\mathbb{R}^2} |\Delta u|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2} |\nabla u|^2|\nabla N * u|^2\,dx\Big)^{1/2}\\ &\le \frac{1}{4}\int_{\mathbb{R}^2} |\Delta u|^2\,dx + \int_{\mathbb{R}^2} |\nabla u|^2|\nabla N * u|^2\,dx,\end{aligned}$$

because $\sqrt{ab} \le (1/4)a + b$ for all $a, b \in [0, \infty)$. Similarly,

$$\int_{\mathbb{R}^2} |\Delta u|u^2\,dx \le \frac{1}{4}\int_{\mathbb{R}^2} |\Delta u|^2\,dx + \int_{\mathbb{R}^2} u^4\,dx.$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} |\Delta u|^2\,dx &\le \frac{1}{4}\int_{\mathbb{R}^2} |\Delta u|^2\,dx + \int_{\mathbb{R}^2} |\nabla u|^2|\nabla N * u|^2\,dx\\ &\quad + \frac{1}{4}\int_{\mathbb{R}^2} |\Delta u|^2\,dx + \int_{\mathbb{R}^2} u^4\,dx\end{aligned}$$

and, consequently,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} |\Delta u(t)|^2\,dx\\ &\qquad \le \int_{\mathbb{R}^2} |\nabla u(t)|^2|(\nabla N * u)(t)|^2\,dx + \int_{\mathbb{R}^2} u^4(t)\,dx.\end{aligned} \tag{5.36}$$

By (5.33) and (5.26), we have that

$$\sup_{t\ge 1} \|(\nabla N * u)(t)\|_\infty < \infty. \tag{5.37}$$

Thus, due to (5.37), we find from Lemma 5.7 that

$$\sup_{t\ge 1}\int_t^{t+1}\int_{\mathbb{R}^2} |\nabla u(s)|^2|(\nabla N * u)(s)|^2\,dxds < \infty.$$

Moreover, (5.26) also implies that

$$\sup_{t\ge 1}\int_t^{t+1}\int_{\mathbb{R}^2} u^4(s, x)\,dxds < \infty.$$

Therefore, (5.34) can be deduced by applying Lemma 5.8 to (5.36).

Finally, by (5.26), (5.34) and (5.37), (5.35) follows from

$$\partial_t u = \Delta u - \langle \nabla u, \nabla N * u\rangle + u^2.$$

The proof is complete. □

Proposition 5.2. *For every* $2 < p < \infty$,

$$\sup_{t\ge 3} \|\nabla u(t)\|_p < \infty.$$

Proof. For $t \geq 3$,

$$\begin{aligned} u(t) &= e^{(t-2)\Delta}u(2) - \int_2^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds \\ &= e^{(t-2)\Delta}u(2) + I(t) + J(t), \end{aligned} \tag{5.38}$$

where

$$\begin{aligned} I(t) &= -\int_2^{t-1} \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds, \\ J(t) &= -\int_{t-1}^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds. \end{aligned}$$

Let $2 < p < \infty$. Applying the L^p-L^q estimates (2.2) with $q = p$ and $m = n = 0$ yields to

$$\|\partial_x e^{(t-2)\Delta}u(2)\|_p \leq C(t-2)^{-1/2}\|u(2)\|_p \qquad \text{for all } t > 2.$$

Thus,

$$\sup_{t\geq 3} \|\partial_x e^{(t-2)\Delta}u(2)\|_p < \infty. \tag{5.39}$$

Now, we estimate $\|\partial_x I(t)\|_p$. Applying (2.2) with $q = 4/3 < 2 < p$, $m = 0$ and $n = 2$, it becomes apparent that there exists $C_1 \geq 0$ such that

$$\begin{aligned} \|\partial_x I(t)\|_p &\leq C_1 \int_2^{t-1} (t-s)^{-3/4+1/p-1}\|u(s)(\nabla N * u)(s)\|_{4/3}\,ds \\ &\leq C_1 \int_2^{t-1} (t-s)^{-3/4+1/p-1}\,ds \sup_{s\geq 2}\|u(s)(\nabla N * u)(s)\|_{4/3} \\ &\leq C_1 \frac{4p}{3p-4} \sup_{s\geq 2}\|u(s)(\nabla N * u)(s)\|_{4/3}. \end{aligned} \tag{5.40}$$

Moreover,

$$\|u(s)(\nabla N * u)(s)\|_{4/3} \leq \|u(s)\|_{4/3}\|(\nabla N * u)(s)\|_\infty$$

and, hence, we conclude from (5.26), (5.37) and (5.40) that

$$\sup_{t\geq 3}\|\partial_x I(t)\|_p < \infty. \tag{5.41}$$

To estimate $\|\partial_x J(t)\|_p$, we first use $\nabla \cdot (\nabla N * u) = -u$ to rewrite $J(t)$ as

$$J(t) = J_1(t) + J_2(t), \tag{5.42}$$

where

$$J_1(t) := -\int_{t-1}^t e^{(t-s)\Delta}\langle \nabla u(s), (\nabla N * u)(s)\rangle\,ds, \quad J_2(t) := \int_{t-1}^t e^{(t-s)\Delta}u^2(s)\,ds.$$

Then, according to (2.2), there exists a constant $C_2 > 0$ such that

$$\begin{aligned}\|\partial_x J_1(t)\|_p &\le C_2 \int_{t-1}^t (t-s)^{-1/2+1/p-1/2} \|\langle \nabla u(s), (\nabla N * u)(s)\rangle\|_2\, ds \\ &\le C_2 \int_{t-1}^t (t-s)^{-1+1/p}\, ds \sup_{s\ge 2} \|\langle \nabla u(s), (\nabla N * u)(s)\rangle\|_2 \\ &\le C_2\, p \sup_{s\ge 2} \|\langle \nabla u(s), (\nabla N * u)(s)\rangle\|_2,\end{aligned}$$

and, consequently, owing to (5.34) and (5.37), we obtain that

$$\sup_{t\ge 3} \|\partial_x J_1(t)\|_p < \infty. \tag{5.43}$$

Similarly, there exists a constant $C_3 > 0$ such that

$$\|\partial_x J_2(t)\|_p \le C_3 \int_{t-1}^t (t-s)^{-1/2+1/p-1/2} \|u^2(s)\|_2\, ds \le C_3 p \sup_{s\ge 2} \|u(s)\|_4^2$$

and, therefore, (5.26) implies that

$$\sup_{t\ge 3} \|\partial_x J_2(t)\|_p < \infty. \tag{5.44}$$

Now, the proof follows easily from (5.38), (5.39), (5.41), (5.42), (5.43) and (5.44). □

5.4. **Convergence to a stationary solution.** This section proves Theorem 5.3. Thus, throughout it, we will assume that the initial data $u_0 \in L^1$ satisfy

$$u_0 \ge 0, \quad \int_{\mathbb{R}^2} u_0\, dx = 8\pi \quad \text{and} \quad \mathcal{H}_b[u_0] < \infty \quad \text{for some} \quad b > 0.$$

By Theorem 5.2, we already know that the unique mild solution u of (KS) is nonnegative and globally defined in time. Moreover,

$$\sup_{t\ge 1} \|u(t)\|_p < \infty \quad \text{for all} \quad 1 \le p \le \infty. \tag{5.45}$$

The proof of Theorem 5.3 will follow after two lemmas of technical nature.

Lemma 5.10. *For every $t > 0$ and $R > 1$ the following uniform integrability estimate holds:*

$$\begin{aligned}\int_{|x|>R} (b+|x|^2)^{1/2} u(t,x)\, dx \le \int_{|x|>R} &(b+|x|^2)^{1/2} w_b(x)\, dx \\ &+ \Phi(b,R)\left(\Psi(b) + \||x|w_b\|_1^{1/2}\right),\end{aligned} \tag{5.46}$$

where

$$\Phi(b,R) := (8b)^{1/4}\mathcal{H}_b[u_0]R^{-1/2}, \quad \Psi(b) := \left(16\pi b^{1/2} + 2(8b)^{1/4}(8\pi)^{1/2}\sqrt{\mathcal{H}_b[u_0]}\right)^{1/2}.$$

Proof. A direct calculation shows that

$$(b+|x|^2)^{1/2} u = (b+|x|^2)^{1/2} w_b + (8b)^{1/4} w_b^{-1/4} (\sqrt{u} - \sqrt{w_b})(\sqrt{u} + \sqrt{w_b}),$$

where $u = u(t,x)$ and $w_b = w_b(x)$. Thus, integrating this identity on $|x| > R$, we have that

$$\int_{|x|>R} (b+|x|^2)^{1/2} u(t,x)\,dx \le \int_{|x|>R} (b+|x|^2)^{1/2} w_b(x)\,dx + (8b)^{1/4} I,$$

where

$$I := \int_{|x|>R} w_b^{-1/4}(x) \left(\sqrt{u(t,x)} - \sqrt{w_b(x)}\right)\left(\sqrt{u(t,x)} + \sqrt{w_b(x)}\right) dx.$$

Using Hölder's inequality and $\mathcal{H}_b[u(t)] \le \mathcal{H}_b[u_0]$ $(t > 0)$ by (5.25), and setting $\Omega := \{|x| > R\}$, we can estimate I as follows.

$$\begin{aligned} I &\le \left(\int_{|x|>R} w_b^{-1/2} (\sqrt{u} - \sqrt{w_b})^2\,dx\right)^{1/2} \left(\int_{|x|>R} (\sqrt{u} + \sqrt{w_b})^2\,dx\right)^{1/2} \\ &\le \mathcal{H}_b[u(t)] \left(\|\sqrt{u} + \sqrt{w_b}\|_{L^2(\Omega)}\right) \le \mathcal{H}_b[u(t)] \left(\|\sqrt{u}\|_{L^2(\Omega)} + \|\sqrt{w_b}\|_{L^2(\Omega)}\right) \\ &\le \mathcal{H}_b[u_0] R^{-1/2} \left[\left(\int_{|x|>R} |x| u(t,x)\,dx\right)^{1/2} + \left(\int_{|x|>R} |x| w_b(x)\,dx\right)^{1/2}\right]. \end{aligned}$$

On the other hand, applying Lemma 5.2(iv) to $u(t)$, using the conservation of mass of u and (5.25), we get

$$\begin{aligned} \int_{\mathbb{R}^2} |x| u(t,x)\,dx &\le 16\pi b^{1/2} + (8b)^{1/4}\left(\|u(t)\|_1^{1/2} + \|w_b\|_1^{1/2}\right)\sqrt{\mathcal{H}_b[u(t)]} \\ &\le 16\pi b^{1/2} + (8b)^{1/4}\left(\|u_0\|_1^{1/2} + \|w_b\|_1^{1/2}\right)\sqrt{\mathcal{H}_b[u_0]} \\ &\le 16\pi b^{1/2} + 2(8b)^{1/4}(8\pi)^{1/2}\sqrt{\mathcal{H}_b[u_0]} \end{aligned}$$

and, therefore,

$$I \le \mathcal{H}_b[u_0] R^{-1/2} \left(\Psi(b) + \||x| w_b\|_1^{1/2}\right).$$

This concludes the proof. □

The next result establishes the averaged large-time asymptotic of the solution.

Lemma 5.11. *For every* $1 \le p \le 2$,

$$\lim_{T\to\infty} \int_T^{T+1} \int_{\mathbb{R}^2} |u(t,x) - w_{b,x_0}(x)|^p\,dxdt = 0, \tag{5.47}$$

where x_0 *is the center of mass of* u_0.

Proof. By the conservation of the center of mass (1.3) and the translational invariance of the problem, we may assume $x_0 = 0$ without lost of generality.

Let $\{t_n\}_{n\ge1}$ be an arbitrary sequence of times such that

$$\lim_{n\to\infty} t_n = \infty$$

and consider the translated solutions

$$u_n(t,x) := u(t+t_n, x), \qquad 0 \le t \le 1,\ x \in \mathbb{R}^2$$

for all $n \geq 1$. According to (5.45) and Proposition 5.1, it is apparent that

$$\sup_{n\geq 1}\sup_{0\leq t\leq 1}\|u_n(t)\|_{H^1} < \infty, \tag{5.48}$$

$$\sup_{n\geq 1}\int_0^1 \|\partial_t u_n(t)\|_2^2\,dt < \infty. \tag{5.49}$$

By the proof of Lemma 5.10, we already know that

$$\sup_{n\geq 1}\sup_{0\leq t\leq 1}\int_{\mathbb{R}^2}|x|u_n(t,x)\,dx \leq \Psi^2(b) < \infty. \tag{5.50}$$

Now, we will show that

$$\{u_n(t)\}_{n=1}^{\infty} \text{ is relatively compact in } L^2(\mathbb{R}^2) \text{ for all } 0\leq t\leq 1. \tag{5.51}$$

Take any $t \in [0,1]$ and fix it. By (5.48), $\{u_n(t)\}_{n\geq 1}$ is bounded in H^1. Thus, by the compactness of the embedding $H^1(B_R) \hookrightarrow L^2(B_R)$, $R > 0$, we can extract a subsequence of $\{u_n(t)\}_{n\geq 1}$, relabeled by $\{u_n(t)\}_{n\geq 1}$, and a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{n\to\infty}\|u_n(t)-v\|_{L^2(B_R)} = 0 \quad \text{for all } R > 0. \tag{5.52}$$

We claim that, actually, $v \in L^2(\mathbb{R}^2)$ and that, along some subsequence,

$$\lim_{n\to\infty}\|u_n(t)-v\|_{L^2(\mathbb{R}^2)} = 0. \tag{5.53}$$

Indeed, by the convergence of $\{u_n(t)\}_{n\geq 1}$ to v in $L^2(B_R)$ for all $R > 0$, we can extract a subsequence, again labeled by n, such that

$$\lim_{n\to\infty} u_n(t,x) = v(x) \qquad a.e. \text{ in } \mathbb{R}^2.$$

As $\{u_n(t)\}_{n\geq 1}$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 \leq p \leq \infty$, we also have $v \in L^p(\mathbb{R}^2)$ for all $1 \leq p \leq \infty$. Due to (5.50),

$$\sup_{n\geq 1}\int_{\mathbb{R}^2}|x|u_n(t,x)\,dx \leq \Psi^2(b) < \infty,$$

and hence, thanks to Fatou's lemma, we find that

$$\int_{\mathbb{R}^2}|x|v(x)\,dx \leq \Psi^2(b) < \infty.$$

Thus,

$$\sup_{n\geq 1}\int_{\mathbb{R}^2}|x||u_n(t,x)-v(x)|\,dx \leq 2\Psi^2(b) < \infty. \tag{5.54}$$

Then, owing to (5.54), we find that for $R > 0$,

$$\begin{aligned}\int_{\mathbb{R}^2} |u_n(t) - v|^2\,dx &= \int_{|x|<R} |u_n(t) - v|^2\,dx + \int_{|x|>R} |u_n(t) - v|^2\,dx \\ &\le \|u_n(t) - v\|^2_{L^2(B_R)} + R^{-1}\int_{|x|>R} |x||u_n(t) - v|^2\,dx \\ &\le \|u_n(t) - v\|^2_{L^2(B_R)} + 2CR^{-1}\int_{|x|>R} |x||u_n(t) - v|\,dx \\ &\le \|u_n(t) - v\|^2_{L^2(B_R)} + 4C\Psi^2(b)R^{-1}\end{aligned}$$

for some nonnegative constant C. By this,

$$\limsup_{n\to\infty} \|u_n(t) - v\|_2^2 \le 4C\Psi^2(b)R^{-1},$$

and then, by letting $R \to \infty$, (5.53) is derived, and hence, (5.51) holds.

Next, owing to (5.49), we obtain that, for any $0 \le t_1 < t_2 \le 1$,

$$\|u_n(t_2) - u_n(t_1)\|_2 \le \int_{t_1}^{t_2} \|\partial_t u_n(t)\|_2\,dt \le |t_2 - t_1|^{1/2} \sup_{n\ge 1}\int_0^1 \|\partial_t u_n(t)\|_2^2\,dt$$

and, therefore,

$$\{u_n\}_{n\ge 1} \quad \text{is uniformly equicontinuos in } \ C([0,1]; L^2).$$

Then, by the Ascoli-Arzela theorem (see, e.g., Lemma 1 of [46]), $\{u_n\}_{n\ge1}$ is relatively compact in $C([0,1]; L^2)$. Therefore, there exists $w \in C([0,1]; L^2)$ and, along some subsequence, relabeled by n, we must have

$$\lim_{n\to\infty} u_n = w \quad \text{in } \ C([0,1]; L^2). \tag{5.55}$$

Actually, from (5.50) it follows that

$$\sup_{0\le t\le 1} \int_{\mathbb{R}^2} |x| w(t,x)\,dx \le \Psi^2(b) < \infty,$$

and from (5.55) it is easily seen that

$$\lim_{n\to\infty} u_n = w \quad \text{in } \ C([0,1]; L^1). \tag{5.56}$$

According to Theorem 5.4,

$$\mathcal{H}_b[u_n(t)] + \int_0^t \mathcal{D}[u_n(s)]\,ds \le \mathcal{H}_b[u_0], \qquad 0 \le t \le 1, \quad n \ge 1.$$

Thus,

$$\begin{aligned}8\int_0^1 \int_{\mathbb{R}^2} |\nabla u_n^{1/4}|^2\,dxdt &= \int_0^1 \mathcal{D}[u_n(t)]\,dt + \int_0^1 \|u_n(t)\|_{3/2}^{3/2}\,dt \\ &\le \mathcal{H}_b[u_0] + \sup_{t\ge 1} \|u(t)\|_{3/2}^{3/2}.\end{aligned} \tag{5.57}$$

By (5.56),

$$\lim_{n\to\infty} u_n^{1/4} = w^{1/4} \qquad \text{in} \quad C([0,1]; L^4).$$

Thus, due to (5.57), we may assume that

$$\lim_{n\to\infty} \nabla u_n^{1/4} = \nabla w^{1/4} \quad \text{weakly in} \quad L^2((0,1)\times\mathbb{R}^2).$$

Hence,

$$\int_0^1 \int_{\mathbb{R}^2} |\nabla w^{1/4}|^2 \, dxdt \le \liminf_{n\to\infty} \int_0^1 \int_{\mathbb{R}^2} |\nabla u_n^{1/4}|^2 \, dxdt$$

and, therefore,

$$\int_0^1 \mathcal{D}[w(t)]\, dt \le \liminf_{n\to\infty} \int_0^1 \mathcal{D}[u_n(t)]\, dt \tag{5.58}$$

(cf. Corollary III.8 of [13], if necessary). Once again by Theorem 5.4, we also find that

$$\int_0^\infty \mathcal{D}[u(t)]\, dt \le \mathcal{H}_b[u_0].$$

Consequently, since

$$\int_0^1 \mathcal{D}[u_n(t)]\, dt = \int_0^1 \mathcal{D}[u(t+t_n)]\, dt = \int_{t_n}^{t_n+1} \mathcal{D}[u(s)]\, ds$$

for all $n \ge 1$, it becomes apparent that

$$\lim_{n\to\infty} \int_0^1 \mathcal{D}[u_n(t)]\, dt = 0.$$

Therefore, (5.58) entails

$$\int_0^1 \mathcal{D}[w(t)]\, dt = 0. \tag{5.59}$$

As, according to Lemma 5.3, we have $\mathcal{D}[w(t)] \ge 0$, the identity (5.59) implies $\mathcal{D}[w(t)] = 0$ for all $t \in [0,1]\setminus N$, where N is a subset of $[0,1]$ of measure zero. Consequently, once again by Lemma 5.3, for every $t \in [0,1]\setminus N$, there exist $b(t) > 0$ and $x_0(t) \in \mathbb{R}^2$ such that

$$w(t) = w_{b(t),x_0(t)}.$$

By (5.46), we observe that

$$\sup_{n\ge 1} \int_{|x|>R} |x| u_n(t,x)\, dx \to 0 \quad \text{as} \quad R \to \infty.$$

Hence, since $u_n(t) \to w(t)$ in L^1 as $n \to \infty$, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} x u_n(t,x)\, dx = \int_{\mathbb{R}^2} x w(t,x)\, dx = \int_{\mathbb{R}^2} x w_{b(t),x_0(t)}(x)\, dx = 8\pi x_0(t).$$

As we are assuming that the center of mass of u_0 is zero, by the conservation of the center of mass for $u(t)$, we have that

$$\int_{\mathbb{R}^2} x u_n(t,x)\, dx = \int_{\mathbb{R}^2} x u_0(x)\, dx = 0.$$

Therefore, $x_0(t) = 0$ and, hence,

$$w(t) = w_{b(t)}, \qquad t \in [0,1]\setminus N.$$

By (5.56), for every $t \in [0,1]\backslash N$, there exists a subsequence $\{u_{n_j}(t)\}_{j\geq 1}$ of $\{u_n(t)\}_{n\geq 1}$ such that

$$\lim_{j\to\infty} u_{n_j}(t,x) = w_{b(t)}(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^2.$$

Then, thanks to Fatou's lemma, (5.25) implies that

$$\begin{aligned}\mathcal{H}_b[w_{b(t)}] &= \int_{\mathbb{R}^2} (\sqrt{w_{b(t)}} - \sqrt{w_b})^2 w_b^{-1/2}\, dx \\ &\leq \liminf_{j\to\infty} \int_{\mathbb{R}^2} \left(\sqrt{u_{n_j}(t)} - \sqrt{w_b}\right)^2 w_b^{-1/2}\, dx \\ &= \liminf_{j\to\infty} \mathcal{H}_b[u_{n_j}(t)] = \liminf_{j\to\infty} \mathcal{H}_b[u(t+t_{n_j})] \leq \mathcal{H}_b[u_0] < \infty.\end{aligned}$$

Consequently, according to Lemma 5.2(i), $b(t) = b$ for all $t \in [0,1]\backslash N$ and, therefore, $w(t) = w_b$ for all $t \in [0,1] \setminus N$. Since $w : [0,1] \to L^1 \cap L^2$ is continuous, we have $w(t) = w_b$ for all $t \in [0,1]$. Moreover, owing to (5.55) and (5.56), we also find that, for every $p = 1, 2$,

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \int_{\mathbb{R}^2} |u(t,x) - w_b(x)|^p\, dxdt = \lim_{n\to\infty} \int_0^1 \int_{\mathbb{R}^2} |u_n(t,x) - w_b(x)|^p\, dxdt = 0.$$

This provides us with (5.47) for $p = 1, 2$. The general case when $1 \leq p \leq 2$ follows from the following interpolation inequality: for every $1 \leq q < p < r \leq \infty$ and $\lambda \in [0,1]$ with $1/p = \lambda/q + (1-\lambda)/r$,

$$\|f\|_p \leq \|f\|_q^\lambda \|f\|_r^{1-\lambda} \quad \text{for all } \; f \in L^q \cap L^p.$$

This ends the proof. □

Proof of Theorem 5.3. As in Lemma 5.11, we may assume that the center of mass of u_0 is zero, that is, $x_0 = 0$. Take any sequence of times $\{t_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} t_n = \infty.$$

Due to Lemma 5.11, we have that

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \|u(t) - w_b\|_2^2\, dt = 0. \tag{5.60}$$

Thus, for every $n \geq 1$, there exists $s_n \in [t_n, t_n+1]$ such that

$$\lim_{n\to\infty} u(s_n) = w_b \quad \text{in } \; L^2.$$

On the other hand, setting

$$I_n := \left| \|u(s_n) - w_b\|_2^2 - \|u(t_n) - w_b\|_2^2 \right|, \qquad n \geq 1,$$

we have that

$$\begin{aligned} I_n &= \left| \int_{t_n}^{s_n} \frac{d}{dt} \|u(t) - w_b\|_2^2\, dt \right| \leq 2 \int_{t_n}^{s_n} \int_{\mathbb{R}^2} |u - w_b||\partial_t u|\, dxdt \\ &\leq \left(\int_{t_n}^{t_n+1} \|u(t) - w_b\|_2^2\, dt \right)^{1/2} \left(\int_{t_n}^{t_n+1} \|\partial_t u(t)\|_2^2\, dt \right)^{1/2}\end{aligned}$$

and, hence, by (5.35) and (5.60), we obtain that $\lim_{n\to\infty} I_n = 0$. Consequently,

$$\lim_{n\to\infty} u(t_n) = w_b \quad \text{in} \;\; L^2$$

and, therefore, as this is valid along any sequence $\{t_n\}_{n\geq 1}$ approximating ∞ as $n \to \infty$, we find that

$$\lim_{t\to\infty} u(t) = w_b \quad \text{in} \;\; L^2.$$

Moreover, thanks to Lemma 5.10, we have that

$$\sup_{t>0} \int_{\mathbb{R}^2} |x| u(t,x)\, dx < \infty$$

and, consequently, we also deduce that

$$\lim_{t\to\infty} u(t) = w_b \qquad \text{in} \;\; L^1.$$

Thus, it becomes apparent from the Nash inequality [42]

$$\|f\|_p \leq C_p \|f\|_1^{1/p} \|\nabla f\|_2^{1-1/p}, \qquad 1 \leq p < \infty,$$

that, for every $p \in [1, \infty)$,

$$\lim_{t\to\infty} u(t) = w_b \quad \text{in} \;\; L^p. \tag{5.61}$$

In the case of $p = \infty$, we will use the interpolation inequality establishing that, for any $2 < q < \infty$, there exists a positive constant C_q, depending only on q, such that

$$\|f\|_\infty \leq C_q \|f\|_q^{1-2/q} \|\nabla f\|_q^{2/q}$$

for all $f \in W^{1,q}(\mathbb{R}^2)$ (see, for example, Theorem 9.3 of [21]). According to it, we find that

$$\|u(t) - w_b\|_\infty \leq C_q \|u(t) - w_b\|_q^{1-2/q} \|\nabla(u(t) - w_b)\|_q^{2/q} \tag{5.62}$$

for all $t \geq 3$ and $q \in (2, \infty)$, and by Proposition 5.2,

$$\sup_{t\geq 3} \|\nabla(u(t) - w_b)\|_q < \infty.$$

Therefore, (5.61) and (5.62) imply (5.61) for $p = \infty$. The proof is complete. $\square$

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