Competing effects in an attraction-repulsion chemotaxis model

Lecture notes from Workshop on Chemotaxis,
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(With thanks to coauthor Zhi-An Wang)

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Talk Outline

Chemotaxis

- Chemotaxis: Biased movement of cells in response to the gradient of a chemical signal

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- Chemotaxisattraction: movement towards higher signal concentration
- Chemotaxisrepulsion: movement towards lower signal concentration
- Interesting phenomenons: Aggregation, Patterning
- Mathematical interests: boundedness, blow-up, existence of non-constant steady states

An attraction-repulsion model

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- LIU/WANG 2011 JBD ($\tau = 1$, 1-D)

The attraction model: Boundedness vs. blow-up

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v - v + u \end{cases}$$

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- $n = 2$: If Ω is a ball, solutions may blow up in finite time (Herrero/Velázquez 1996 Pisa)
- $n \geq 3$: If Ω is a ball, then *for all* $M > 0$ $\exists$ initial data with $\int_\Omega u_0 = M$ such that (u, v) blows up in finite time (Winkler 2012 preprint)

The attraction model: Key technical ingredients

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ v_t = \Delta v - v + u \end{cases}$$

- $n = 2$: The proof of 'critical mass' strongly depends on the Lyapunov functional

$$\mathcal{F}(u, v) := \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u$$

and the Moser-truding inequality

$$\int_\Omega \exp(|\psi|) \le Cexp\Big(|\Omega|^{-1} \Big| \int_\Omega \psi \Big| + \|\nabla\|_2^2/(8\pi) \Big) \qquad \text{for all } \psi \in H^1(\Omega)$$

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- $n \geq 3$: Winkler's proof of the finite-time blow-up also strongly depends on the Lyapunov functional $\mathcal{F}(u, v)$

The simplified attraction model: blow-up

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \\ 0 = \Delta v - v + u \end{cases}$$

- $n = 2$: If Ω is a ball and $0 = \Delta v - M + u, \quad M := \bar{u}_0, \int_\Omega v = 0 \Rightarrow$ finite-time blow-up (JÄGER/LUCKHAUS 1992 TAMS)

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- $n = 2$: Let $O \in \Omega$. $\int_\Omega u_0 > 8\pi$, $\int_\Omega u_0(x)|x|^2$ small $\Rightarrow$ finite-time blow-up (NAGAI 2001 JIA). Moment method: $\int_\Omega u(x,t)\Phi(x)$. The Green function $G(x,y)$ of $-\Delta + 1$ on Ω with zero Neumann boundary condition satisfies (Ito's book: Diffusion Equations)

$$|G(x,y)| \le C\Big(1 + \ln^+ \frac{1}{|x-y|}\Big), \qquad |\nabla_x G(x,y)| \le \frac{C}{|x-y|} \quad ?$$

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The attraction model: Stationary solutions

- $$\begin{cases} D_1\Delta u - \chi\nabla\cdot(u\nabla v) = 0, & x\in\Omega \\ D_2\Delta v - av + bu = 0, & x\in\Omega \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega \end{cases}$$

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where $\epsilon := D_2/a,\ \lambda := bc/a, \beta := \chi/D_1$.

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- $n = 2$: If ε is small, then there exists a positive nonconstant solution v_ε (Kabeya/Ni 1998 RIMS)

The repulsion model: Global existence and convergence

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla v) \\ v_t = \Delta v - v + u \end{cases}$$

- $n = 1, 2$: Global solutions exists and converges to the unique stationary solution $(\bar{u}_0, \bar{u}_0)$ exponentially as $t \to +\infty$ (CIEŚLAK/LAURENÇOT/MORALES-RODRIGO 2008 BCP)

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- $n = 3, 4$: Global existence of weak solutions (Cieślak/Laurençot/Morales-Rodrigo 2008 BCP)
- The analysis strongly depends on

$$\mathcal{F}(u, v) := \int_\Omega \left(u \ln u + \frac{|\nabla v|^2}{2} \right)$$

Attraction-repulsion chemotaxis model: Main difficulty

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) \\
\tau v_t = \Delta v + \alpha u - \beta v \\
\tau w_t = \Delta w + \gamma u - \delta w
\end{cases}
$$

If $\beta \neq \delta$, then there is NOT any Lyapunov functional available for our purpose.

Simplified attraction-repulsion model: Boundedness

- $$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) \\ 0 = \Delta v + \alpha u - \beta v \\ 0 = \Delta w + \gamma u - \delta w \end{cases}$$

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- **THEOREM 1** (T/WANG 2012 M3AS)
  Assume that $0 \leq u_0 \in W^{1,\infty}(\Omega)$ and that $\xi\gamma - \chi\alpha > 0$. Then for any $n \geq 2$, the model admits a unique nonnegative bounded classical solution.

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  Assume that $0 \le u_0 \in W^{1,\infty}(\Omega)$ and that $\xi\gamma - \chi\alpha > 0$. Then for any $n \ge 2$, the model admits a unique nonnegative bounded classical solution.

- $z := \xi w - \chi v \Rightarrow$

$$\begin{cases} u_t = \Delta u + \nabla\cdot(u\nabla z) \\ 0 = \Delta z + (\xi\gamma - \chi\alpha)u - \beta z + \xi(\beta - \delta)w \\ 0 = \Delta w + \gamma u - \delta w \end{cases}$$

Key Lemma

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- PROOF

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}\Delta u - \int_\Omega u^{p-1}\nabla\cdot(\chi u\nabla v) + \int_\Omega u^{p-1}\nabla\cdot(\xi u\nabla w) \\
&= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \chi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\
&\qquad -\xi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla w \\
&= -\frac{4(p-1)}{p^2}\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + \frac{\chi(p-1)}{p}\int_\Omega \nabla u^p\cdot\nabla v - \frac{\xi(p-1)}{p}\int_\Omega \nabla u^p \\
&= -\frac{4(p-1)}{p^2}\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + \frac{p-1}{p}\cdot\int_\Omega u^p(-\chi\Delta v + \xi\Delta w) \\
&= -\frac{4(p-1)}{p^2}\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + \frac{p-1}{p}\cdot\int_\Omega u^p[\xi\delta w - (\xi\gamma - \chi\alpha)u - \chi\beta v]
\end{aligned}$$

Proof of Lemma 1.1: Continue 1

- $$\begin{aligned}\frac{d}{dt}\int_\Omega u^p &\le -(\xi\gamma-\chi\alpha)(p-1)\int_\Omega u^{p+1}+\xi\delta(p-1)\int_\Omega u^p w\\ &\le -\frac{(\xi\gamma-\chi\alpha)(p-1)}{2}\int_\Omega u^{p+1}+c_1\int_\Omega w^{p+1}\end{aligned}$$

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- $$\begin{cases}-\Delta w+\delta w=\gamma u, & x\in\Omega\\ \frac{\partial w}{\partial\nu}=0, & x\in\partial\Omega\end{cases}\quad\Rightarrow \|w(\cdot,t)\|_{W^{2,p}(\Omega)}\le c_2\|u(\cdot,t)\|_{L^p(\Omega)}$$

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- $$\begin{aligned} \int_\Omega w^{p+1}=\|w\|^{p+1}_{L^{p+1}(\Omega)} &\le c_3\|D^2w\|^{(p+1)\theta}_{L^p(\Omega)}\|w\|^{(p+1)(1-\theta)}_{L^1(\Omega)}+c_3\|w\|^{p+1}_{L^1(\Omega)} \\ &\le c_4\|u\|^{(p+1)\theta}_{L^p(\Omega)}+c_4 \end{aligned}$$

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  $$\le c_4\|u\|^{(p+1)\theta}_{L^p(\Omega)} + c_4$$

- $\theta\in(0,1)$, $(p+1)\theta < p \Rightarrow \int_\Omega w^{p+1} \le c_4\int_\Omega u^p + c_4 \le \epsilon c_4\int_\Omega u^{p+1} + c_5(\epsilon)$

Proof of Lemma 1.1: Continue 2

- $$\frac{d}{dt}\int_\Omega u^p \le -\frac{(\xi\gamma-\chi\alpha)(p-1)}{4}\int_\Omega u^{p+1}+c_6$$

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- $$\int_\Omega u^p \le c_8$$

Full 2-D attraction-repulsion model for $\xi\gamma - \chi\alpha > 0$

- $$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) \\ v_t = \Delta v + \alpha u - \beta v \\ w_t = \Delta w + \gamma u - \delta w \end{cases}$$

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- $$\|f\|_{L^4(\Omega)}^4 \le C_{GN}(\|\nabla f\|_{L^2(\Omega)}^2 \cdot \|f\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^4)$$

Full 2-D attraction-repulsion model for $\xi\gamma - \chi\alpha > 0$

- $$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) \\ v_t = \Delta v + \alpha u - \beta v \\ w_t = \Delta w + \gamma u - \delta w \end{cases}$$

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- **THEOREM 2** (T/WANG 2012 M3AS)

  Assume that $0 \le u_0, v_0, w_0 \in W^{1,\infty}(\Omega)$, that $\xi\gamma - \chi\alpha > 0$, and that

  $$\int_\Omega u_0 \le \frac{8\beta^2(\xi\gamma - \chi\alpha)}{C_{GN}\chi^2\alpha^2(\beta-\delta)^2}.$$

  Then, for $n = 2$, the model admits a unique nonnegative locally bounded classical solution.

Entropy-type inequality

- $s := \xi w - \chi v \Rightarrow$

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla s) & \times \ln u \\ s_t = \Delta s + (\xi\gamma - \chi\alpha) u - \delta s + \chi(\beta - \delta) v & \times \left(-\frac{1}{\xi\gamma - \chi\alpha} \Delta s \right) \\ v_t = \Delta v + \alpha u - \beta v & \times v \end{cases}$$

Entropy-type inequality

- $s := \xi w - \chi v \Rightarrow$

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- **LEMMA 2.1**

$$\begin{aligned} \frac{d}{dt}\left\{ \int_\Omega u \ln u \right. & + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_\Omega |\nabla s|^2 + \frac{\chi^2(\beta-\delta)^2}{2\beta(\xi\gamma - \chi\alpha)} \int_\Omega v^2 \left. \right\} \\ & + 4\int_\Omega |\nabla u^{\frac{1}{2}}|^2 + \frac{1}{2(\xi\gamma - \chi\alpha)} \int_\Omega |\Delta s|^2 \\ & \le \frac{\chi^2\alpha^2(\beta-\delta)^2}{2\beta^2(\xi\gamma - \chi\alpha)} \int_\Omega u^2 \end{aligned}$$

Estimate on $\int_0^t \int_\Omega |\nabla s|^4$

- **LEMMA 2.2** For all $t \in (0, T)$, there hold:

$$\int_\Omega |\nabla s(\cdot, t)|^2 \le c(T)$$

$$\int_0^t \int_\Omega |\Delta s|^2 \le c(T)$$

$$\int_0^t \int_\Omega |\nabla s|^4 \le c(T)$$

Estimate on $\int_0^t \int_\Omega |\nabla s|^4$

- **LEMMA 2.2** For all $t \in (0, T)$, there hold:

$$\int_\Omega |\nabla s(\cdot, t)|^2 \leq c(T)$$

$$\int_0^t \int_\Omega |\Delta s|^2 \leq c(T)$$

$$\int_0^t \int_\Omega |\nabla s|^4 \leq c(T)$$

- PROOF

$$\begin{aligned}\int_\Omega u^2 = \|u^{\frac{1}{2}}\|^4_{L^4(\Omega)} &\leq C_{GN}\|\nabla u^{\frac{1}{2}}\|^2_{L^2(\Omega)}\|u^{\frac{1}{2}}\|^2_{L^2(\Omega)} + C_{GN}\|u^{\frac{1}{2}}\|^4_{L^2(\Omega)} \\ &= C_{GN}\|u_0\|_{L^1(\Omega)}\|\nabla u^{\frac{1}{2}}\|^2_{L^2(\Omega)} + C_{GN}\|u_0\|^2_{L^1(\Omega)}\end{aligned}$$

Estimate on $\int_\Omega u^p \ (p > 1)$

LEMMA 2.3 $\int_\Omega u^p \le c(T)$

PROOF

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 \le c_1 \int_\Omega u^p |\nabla s|^2$$

$$\int_\Omega u^p |\nabla s|^2 \le \Big(\int_\Omega u^{2p}\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\nabla s|^4\Big)^{\frac{1}{2}}$$

$$\Big(\int_\Omega u^{2p}\Big)^{\frac{1}{2}} = \|u^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le c_2\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} \cdot \|u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)}\Big)$$

$$\|f\|^2_{L^2} \le \|f\|^{4b}_{L^4} \cdot \|f\|^{\frac{2(1-b)}{p}}_{L^{\frac{2}{p}}}, \quad b := (p-1)/(2p-1) \Rightarrow \|f\|^2_{L^2} \le \varepsilon\|f\|^2_{L^4} + c(\varepsilon)\|f\|^2_{L^{\frac{2}{p}}}$$

$$\|u^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le c_3\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} \cdot \|u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\Big)$$

$$c_1 \int_\Omega u^p |\nabla s|^2 \le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + c_4\Big(\int_\Omega |\nabla s|^4\Big) \cdot \Big(\int_\Omega u^p + 1\Big)$$

$$y'(t) \le c_5\Big(\int_\Omega |\nabla s|^4\Big) \cdot (y(t)+1) \Rightarrow y(t) + 1 \le (y(0)+1) \cdot e^{c_5 \int_0^t \int_\Omega |\nabla s|^4}$$

Open problem 1: Blow-up?

Q 1. Let $n \geq 3$ and $\chi\alpha - \xi\gamma > 0$, $\beta \neq \delta$.
For any $M > 0$, to prove that there exists some (u_0, v_0, w_0) with $\int_\Omega u_0 = M$ such that the corresponding solution blows up in a finite time.

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) \\
v_t = \Delta v + \alpha u - \beta v \\
w_t = \Delta w + \gamma u - \delta w \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \\
u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x)
\end{cases}
$$

Open problem 2: Nonconstant stationary solution?

- **Q 2.** Let $n = 2$ and $\chi\alpha - \xi\gamma > 0$, $\beta \neq \delta$.
  To prove the existence of nonconstant stationary solutions.

$$\begin{cases} \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) = 0 \\ \Delta v + \alpha u - \beta v = 0 \\ \Delta w + \gamma u - \delta w = 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \\ u > 0, \quad v > 0, \quad w > 0 \end{cases}$$

Open problem 2: Nonconstant stationary solution?

- **Q 2.** Let $n=2$ and $\chi\alpha-\xi\gamma>0$, $\beta\neq\delta$.
  To prove the existence of nonconstant stationary solutions.

$$\begin{cases} \Delta u-\nabla\cdot(\chi u\nabla v)+\nabla\cdot(\xi u\nabla w)=0 \\ \Delta v+\alpha u-\beta v=0 \\ \Delta w+\gamma u-\delta w=0 \\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial w}{\partial\nu}=0 \\ u>0, \quad v>0, \quad w>0 \end{cases}$$

- $u=ce^{\chi v-\xi w}$, $z:=\chi v-\xi w \Rightarrow$

$$\begin{cases} \Delta z+(\chi\alpha-\xi\gamma)ce^{z}-\beta z+\xi(\delta-\beta)w=0 \\ \Delta w+\gamma ce^{z}-\delta w=0 \\ \frac{\partial z}{\partial\nu}=\frac{\partial w}{\partial\nu}=0 \\ z>0, \quad w>0 \end{cases}$$

Thank You

- Thanks to you $:-)$