Recent Mathematical Development on Chemotaxis Models with Logarithmic Chemotactic Sensitivity

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Recent mathematical development on chemotaxis models with logarithmic sensitivity

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Directed movement

Characteristic feature of living organisms: sense and response by adjusting the direction of movement to external stimuli, such a response is called "taxis". If the external stimulus is due to chemical gradient, it is called chemotaxis. Other examples: geotaxis, thermotaxis, phototaxis, and so on.

1. In the last 30-35 years, chemotaxis covers $> 95\%$ among all works related to taxes (e.g. phototaxis, thermotaxis, geotaxis, electrotaxis)
2. 1975-2006, more than 22000 papers (PubMed) on chemotaxis were published

Type of chemotaxis:

1. attractive chemotaxis: if movement is toward the external stimuli;
2. repulsive chemotaxis: if movement is away from the external stimuli.

Convention: attractive chemotaxis is commonly referred to as chemotaxis.

Chemotaxis

Chemotaxis of Neutrophils chasing bacterial

Chemotaxis of a population of Dictyostelium cells to cAMP

Applications: finding food, avoid poisons (predator), embryo development, wound healing, immune response, cancer metastasis, and more.

Chemotaxis model

Prototype chemotaxis model is due to Keller-Segel (1971)

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla \phi(v)) \\ \frac{\partial v}{\partial t} = \varepsilon \Delta v + k(u, v) \end{cases}$$

u - cell density;
v - chemical concentration;
χ - chemotactic coefficient;
$\phi(v)$ - chemotactic sensitivity (potential) function;
ε - chemical diffusion coefficients;

Forms of $\phi(v)$ frequently used:

1. Linear law: $\phi(v) = v$;
2. Receptor law: $\phi(v) = \frac{kv^m}{1+v^m}, k > 0, m \in \mathbb{Z}^+$;
3. Logarithmic law: $\phi(v) = \log v$;

More forms can be found in a paper by "M.J. Tindall, P.K. Maini, S.L. Porter, and J.P. Armitage, Bull. Math. Biol., 70:1570-1607, 2008".

Outline

Chemotaxis models with logarithmic sensitivity

1. Experimental evidence (measurement) for logarithmic sensitivity;
2. Introduction to chemotaxis models with logarithmic sensitivity;
3. Traveling waves: existence, wave speed and stability;
4. Well-posedness: global existence, large-time behavior and stationary solutions;

The Weber-Fechner law in chemotaxis

The Weber-Fechner law: attempts to describe the relationship between the physical magnitudes of stimuli and the perceived intensity of the stimuli, and asserts that the relationship between perception and stimuli is logarithmic:

$$p = k \log \frac{S}{S_0}$$

where p is the perception, log is the natural logarithm, and S_0 is the threshold of stimulus below which it is not perceived at all, constant factor k is to be determined experimentally.

Application in chemotaxis $\phi(v) = k \log \frac{v}{v_0}$ leads to

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi \frac{u}{v} \nabla v) \\ \frac{\partial v}{\partial t} = \varepsilon \Delta v + k(u, v) \end{cases}$$

Weber-Fechner law in bacteria chemotaxis

J. Adler, Chemotaxis in bacteria, Science, 153(1966): 708-716.

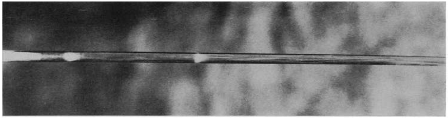


Fig. 1. Photograph showing bands of *E. coli* in a capillary tube. In all the experiments reported here, capillary tubes (*18*) were filled with a liquid medium (*19*), inoculated at one end with 2×10^5 to 2×10^6 bacteria (*20*), and then closed at the ends with plugs of agar and clay, all according to a procedure described in full elsewhere (*8*). The tubes were incubated horizontally at 37°C. The origin, which is turbid because of the bacteria that have not moved out, is visible at the left, then the second band of bacteria, then the first band. Plugs at ends are not shown. The concentration of galactose was 2.5×10^{-4} mole per liter.

E.F. Keller, L.A. Segel, Traveling bands of chemotaxis bacteria: a theoretical analysis, J. Theor. Biol., 30(1971): 235-248.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d\frac{\partial u}{\partial x} - \chi \frac{u}{v}\frac{\partial v}{\partial x}\right) \\ \frac{\partial v}{\partial t} = \varepsilon \frac{\partial^2 v}{\partial x^2} - uv^m \end{cases}$$

u - bacteria density;
v - oxygen concentration;
χ - chemotactic coefficient;
m - consumption rate

Note. In above model, it has been shown that any sensitivity function other than logarithmic law will not reproduce the observed pulse propagation phenomenon.

Logarithmic law in reinforced random walk

Othmer and Stevens, "Aggregation, blowup and collapse: The ABC's of taxis in reinforced random walk", SIAM J. Appl. Math., 57: 1044-1081, 1997.

$$\frac{\partial u_i}{\partial t} = \mathcal{T}^+_{i-1} u_{i-1} + \mathcal{T}^-_{i+1} u_{i+1} - (\mathcal{T}^+_i + \mathcal{T}^-_i) u_i,$$

where $\mathcal{T}^\pm_i$ are the transitional rate per unit of time for a one-step jump to $i \pm 1$ with jump distance h.
Assumption (barrier model): the chemical v-dependence of the transition at site i is localized at $i \pm 1/2$ and when to jump is made independently of the decision where to jump (i.e. the mean waiting time $(\mathcal{T}^+_i + \mathcal{T}^-_i)^{-1}$ is constant given by $1/(2\lambda)$), and

$$\mathcal{T}^\pm_i = \mathcal{T}^\pm_i(v) = 2\lambda \frac{v_{i\pm 1/2}}{v_{i-1/2} + v_{i+1/2}}.$$

Then applying the Taylor expansion, using $x = ih$ and taking the limit as $h \to 0, \lambda \to \infty$ such that $D = \lim_{h\to 0, \lambda\to\infty} \lambda h^2$, one obtains that

$$\frac{\partial u}{\partial t} = D \frac{\partial}{\partial x}\left(u \frac{\partial}{\partial x}\Big(\log \frac{u}{v} \Big) \right) = D \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} - \frac{u}{v}\frac{\partial v}{\partial x} \Big)$$

Experiment measurement of logarithmic sensitivity

The experiment measurements corroborating logarithmic sensitivity:

1. Yevgeniy V. Kalinin *et al*, Logarithmic sensing in *Escherichis coli* bacterial chemotaxis, Biophy. J, 96: 2439-2448, 2009.

   "By measuring the E. coli cell density profiles across the microfluidic channel at various spatial gradients of ligand concentration grad[L] and the average ligand concentration $[\bar{L}]$ near the peak chemotactic response region, we demonstrated unambiguously in both experiments and model simulation that the mean chemotactic drift velocity of E. coli cells increased monotonically with grad(log L), that is E. coli cells sense the spatial gradient of the logarithmic ligand concentration".

2. Yuhai Tu *et al*, Modeling the chemotactic response of *Escherichis coli* to time-varying stimuli, PNAS, 105: 14855-14860, 2008

   and

   Melissia Reneaux *et al*, Theoretical results for chemotactic response and drift of *Escherichis coli* in a weak attractant gradient, J. Theor. Biol., 266:99-106, 2010.

   "Logarithmic model testing the experiment data"

Some chemotaxis models with logarithmic sensitivity

1. Reinforced random walk model (Othmer-Stevens model)

$$\begin{cases} u_t = (Du_x - \chi u \frac{v_x}{v})_x, \\ v_t = \varepsilon v_{xx} + uv^{\gamma} - \beta v \end{cases} \quad (1)$$

   where $\chi \in \mathbb{R}/\{0\}$, and

   $\gamma = 0$ (linear growth for the chemical)

   or

   $\gamma = 1$ (exponential growth).

2. Bacterial chemotaxis model (Keller-Segel model)

$$\begin{cases} u_t = (Du_x - \chi u \frac{v_x}{v})_x, \\ v_t = \varepsilon v_{xx} - uv^{m} \end{cases} \quad (2)$$

Reinforced random walk model with exponential growth

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \frac{\nabla v}{v}), \\ v_t = \varepsilon \Delta v + uv - \beta v \end{cases}$$

with $\chi < 0$ (repulsive chemotaxis) or $\chi > 0$ (attractive chemotaxis)

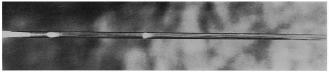
1. Traveling wave solutions: existence, wave speed, stability
2. Initial boundary value problem (IBVP): global regularity
3. Cauchy problem: global regularity

Key idea: $\mathbf{w} = -\frac{\nabla v}{v}$ leads to a system of conservation law without logarithm singularity (Li and Wang, 2010)

$$\begin{cases} u_t - \chi \nabla \cdot (u\mathbf{w}) = D\Delta u, \\ \mathbf{w}_t - \nabla(\varepsilon \mathbf{w}^2 + u) = \varepsilon \Delta \mathbf{w}. \end{cases}$$

Traveling bands of bacterial chemotaxis

J. Adler, Chemotaxis in bacteria, Science, 153(1966): 708-716.



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E.F. Keller, L.A. Segel, Traveling bands of chemotaxis bacteria: a theoretical analysis, J. Theor. Biol., 30(1971): 235-248.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d\frac{\partial u}{\partial x} - \chi \frac{u}{v}\frac{\partial v}{\partial x}\right) \\ \frac{\partial v}{\partial t} = \varepsilon \frac{\partial^2 v}{\partial x^2} - uv^m \end{cases}$$

u: bacterial density;
v: concentration of oxygen
$d > 0, \chi > 0, \varepsilon \geq 0, m \geq 0$

1. $\varepsilon = 0$ and $0 \leq m < 1$: existence and stability of traveling wave solution (t.w.s) by Keller and Segel, Rosen, Odell, and others (1971-1984).
2. $\varepsilon > 0$ and $m = 0$: existence and diffusion limits by Nagai and Ikeda(1991) and extended by Lui & Wang (2010)
3. $\varepsilon > 0$ and $0 \leq m < 1$: existence by Schwetlick(2003), and by Yong Li (2004)

Open question: Stability (linear or nonlinear) of tws for $\varepsilon \geq 0, 0 < m < 1$?

Traveling wave of chemotaxis model with log sensitivity

Models considered:

1. Repulsive RRW model

$$\begin{cases} u_t = (Du_x - \chi u \frac{v_x}{v})_x, & x \in \mathbb{R}, t > 0 \\ v_t = \varepsilon v_{xx} + uv - \beta v \end{cases} \tag{3}$$

   with $\beta \geq 0$ with $\chi < 0$.

2. Angiogenesis model

$$\begin{cases} u_t = (Du_x - \chi u v^{-\alpha} v_x)_x, & x \in \mathbb{R}, t > 0 \\ v_t = \varepsilon v_{xx} - uv \end{cases} \tag{4}$$

   where $\chi > 0$ and $\alpha = 1$.

Challenges:

1. high dimensionality
2. singularity

Methods:

1. reduction to a lower dimensional system
2. removal of singularity

Hopf-Cole transformation

1. Repulsive RRW model ($\chi < 0$)

$$\begin{cases} u_t = (Du_x - \chi u \frac{v_x}{v})_x, \\ v_t = \varepsilon v_{xx} + uv - \beta v \end{cases} \overset{w=\frac{v_x}{v}}{\Longrightarrow} \begin{cases} u_t - \tilde{\chi}(uw)_x = Du_{xx}, \\ w_t + (-\varepsilon w^2 - u)_x = \varepsilon w_{xx} \end{cases} \tag{5}$$

where $\tilde{\chi} = -\chi > 0$.

2. Angiogenesis model ($\chi > 0$)

$$\begin{cases} u_t = (Du_x - \chi uv^{-1} v_x)_x, \\ v_t = \varepsilon v_{xx} - uv \end{cases} \overset{w=-\frac{v_x}{v}}{\Longrightarrow} \begin{cases} u_t - \chi(uw)_x = Du_{xx}, \\ w_t + (\varepsilon w^2 - u)_x = \varepsilon w_{xx} \end{cases} \tag{6}$$

Note: When $\varepsilon = 0$, (5)=(6).

TWS for transformed system with non-diffusive chemical

Setting $\varepsilon = 0$, the model becomes

$$\begin{cases} u_t - \chi(uw)_x = Du_{xx}, \\ w_t - u_x = 0 \end{cases}$$

Traveling waves: Consider a particular solution in a moving coordinate system $z = x - ct \in \mathbb{R}$ with constant speed c. Then the above system transforms into

$$\begin{cases} u_t = Du_{zz} + \chi(uw)_z + cu_z, \\ w_t = u_z + cw_z \end{cases} \tag{7}$$

Then a traveling wave is a stationary solution, denoted by $(U(z), W(z))$, of (7) in the space $C^2(\mathbb{R})^2$ of functions converging at $\pm\infty$, which satisfies $(' = d/dz)$,

$$\begin{cases} DU'' + \chi(UW)' + cU' = 0, \\ U' + cW' = 0 \end{cases}$$

with $(U, W)(z) \to (u_\pm, w_\pm)$ as $z \to \pm\infty$, where $u_\pm \geq 0, w_\pm \leq 0$.

Traveling waves of transformed system

Theorem: Let $u_\pm \geq 0, v_\pm \leq 0$. Then

(i) [existence, Wang and Hillen 2007, Math. Methods. Appl. Sci.] Monotonic wavefronts $(U, W)(x - ct)$ with $U' < 0$ and $W' > 0$ exist for any $\chi > 0, D > 0$, which is unique up to a translation, where

$$c = -\frac{\chi w_-}{2} + \frac{1}{2}\sqrt{\chi^2 w_-^2 + 4\chi u_+}$$

(ii) [nonlinear stability, Li and Wang 2009, SIAM J. Appl. Math.] If $u_+ > 0$, then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{H^1(\mathbb{R})} + \|w_0 - W\|_{H^1(\mathbb{R})} + \|(\phi_0, \psi_0)\|_{L^2(\mathbb{R})} \leq \varepsilon_0$, there is a unique global solution $(u, w)(x, t)$ such that $(u - U, w - W) \in (C([0, \infty); H^1(\mathbb{R})) \cap L^2([0, \infty); H^1(\mathbb{R})))^2$ with

$$\sup_{x \in \mathbb{R}} |(u, w)(x, t) - (U, W)(x - ct)| \to 0 \quad as \quad t \to \infty.$$

where (u_0, w_0) is the initial date and (ϕ_0, ψ_0) is the initial perturbation

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0 - U, w_0 - W)(y) dy.$$

satisfying $(\phi_0, \psi_0)(\pm\infty) = 0$.

Proof of stability

Main idea: Due to the conservation, the perturbation is made by using the anti-derivative technique

$$(\phi(x,t),\psi(x,t)) = \int_{-\infty}^{x} (u(y,t) - U(y-ct), w(y,t) - W(y-ct))dy$$

with $\phi(\pm\infty, t) = 0,\ \psi(\pm\infty, t) = 0$ for all $t > 0$. Then ($\chi = 1$ after scaling)

$$\begin{cases} \phi_t = D\phi_{zz} + c\phi_z + U\psi_z + W\phi_z + \phi_z\psi_z, \\ \psi_t = c\psi_z + \phi_z. \end{cases}$$

By energy estimates, to get rid of the cross term containing lower order derivatives and obtain the L^2-estimates, we multiply the first equation by $\frac{\phi}{U}$ which requires that U must be strictly positive. Since $u_+ \le U \le u_-$, we impose the condition $u_+ > 0$.

Note: This technique fails for $u_+ = 0$, even in the exponentially weighted space.

Open questions

1. Large perturbation;
2. Convergence rate (exponential decay rates can be excluded)
3. Stability for $u_+ = 0$.

Progress made for the stability of tws with $u_+ = 0$ (with H.Y. Jin)

- Linear stability analysis for $x \in (-\infty, \infty)$

  (1) $\sigma_{\text{point}}(\mathcal{L}) \subset \{\lambda \mid \mathrm{Re}\lambda < 0\}$ and $\sigma_{\text{ess}}(\mathcal{L}) \subset \{\lambda \mid \mathrm{Re}\lambda \leq 0\} \to$ no linear exponential stability, where $\mathcal{L}$ is the linearized operator;

  (2) Instability in exponentially weighted space;
- Nonlinear stability for any small perturbation made in $x \in (-\infty, L)$ for any $L > 0$ (energy method)

Traveling waves with chemical diffusion

1. Repulsive RRW model ($\chi < 0$)

$$\begin{cases} u_t = (Du_x - \chi u \frac{v_x}{v})_x, \\ v_t = \varepsilon v_{xx} + uv - \beta v \end{cases} \xrightarrow{w=\frac{v_x}{v}} \begin{cases} u_t - \tilde{\chi}(uw)_x = Du_{xx}, \quad \tilde{\chi} = -\chi > 0 \\ w_t + (-\varepsilon w^2 - u)_x = \varepsilon w_{xx}. \end{cases}$$

2. Angiogenesis model ($\chi > 0$)

$$\begin{cases} u_t = (Du_x - \chi u v^{-1} v_x)_x, \\ v_t = \varepsilon v_{xx} - uv \end{cases} \xrightarrow{w=-\frac{v_x}{v}} \begin{cases} u_t - \chi(uw)_x = Du_{xx}, \\ w_t + (\varepsilon w^2 - u)_x = \varepsilon w_{xx} \end{cases} \tag{8}$$

Then the traveling wave solutions $(U, V)(z) = (u, v)(x - ct)$ of (8) satisfies

$$\begin{cases} DU'' + \chi(UW)' + cU' = 0, \\ \varepsilon W'' + cW' - (\varepsilon W^2 - U)' = 0 \end{cases}$$

with $(U, W)(z) \to (u_\pm, w_\pm)$ as $z \to \pm\infty$. Then

$$\begin{cases} DU' = -cU - UW + \rho_1, \\ \varepsilon W' = -sW + \varepsilon W^2 - U + \rho_2 \end{cases}$$

where

$$\rho_1 = cu_- + u_- w_- = cu_+ + u_+ w_+,$$

$$\rho_2 = cw_- - \varepsilon(w_-)^2 + u_- = cw_+ - \varepsilon(w_+)^2 + u_+$$

Existence and Stability

Theorem [Li and Wang 2011, JDE]: Let $u_\pm \geq 0, w_\pm \leq 0$. Then

(i) Monotonic wavefronts $(U, W)(x - ct)$ with $U' < 0$ and $W' > 0$ exist for any $\chi > 0, D > 0$, which is unique up to a translation, where

$$c_\varepsilon = -\frac{\chi w_-}{2} + \frac{\chi}{2}\sqrt{w_-^2 + 4u_+\left(1 - \varepsilon\frac{w_+^2 - w_-^2}{u_+ - u_-}\right)}, \ \varepsilon \geq 0$$

(ii) If $u_+ > 0$ and $\varepsilon > 0$ is small, then there exists a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{H^1(\mathbb{R})} + \|w_0 - W\|_{H^1(\mathbb{R})} + \|(\phi_0, \psi_0)\|_{L^2(\mathbb{R})} \leq \varepsilon_0$, there is a unique global solution $(u, w)(x, t)$ such that
$(u - U, w - W) \in (C([0, \infty); H^1(\mathbb{R})) \cap L^2([0, \infty); H^1(\mathbb{R})))^2$ with

$$\sup_{x \in \mathbb{R}} |(u, w)(x, t) - (U, W)(x - ct)| \to 0 \ \ as \ \ t \to \infty.$$

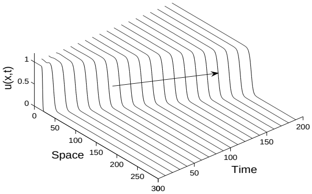
where (u_0, w_0) is the initial date and (ϕ_0, ψ_0) is the initial perturbation

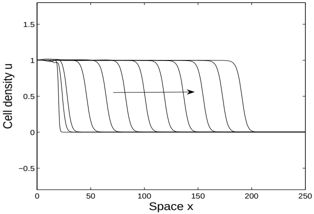
$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (u_0 - U, w_0 - W)(y)dy$$

satisfying $(\phi_0, \psi_0)(\pm\infty) = 0$.

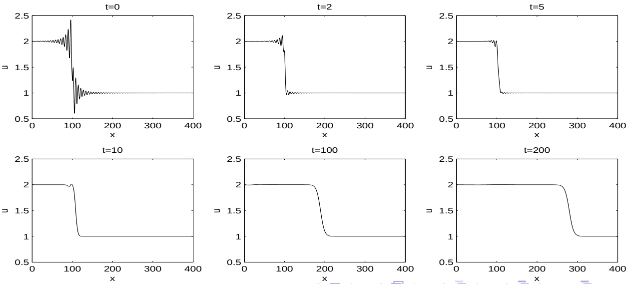
Numerical simulations

Wave Propagation





Stability of Wave Propagation



Passing results back to the original system

By $w = -\frac{v_x}{v}$, we have

Theorem: Let $u_\pm \geq 0, v_\pm \geq 0$. Then

(i) there is unique monotone traveling wavefront solution $(U, V)(x - ct)$ for any $\chi > 0, D > 0$ with $U' < 0, V' > 0$ such that $v_+ > v_- = 0, u_- > u_+ \geq 0$ provided that $\varepsilon \geq 0$ is small, where the wave speed c is given by

$$c = \chi \left[\frac{u_-}{\chi + \varepsilon(1 - u_+/u_-)} \right]^{1/2}.$$

(ii) if $u_+ > 0$ and $\varepsilon \geq 0$ is small, t.w.s is asymptotically stable:

$$\sup_{x \in \mathbb{R}} |(u, v)(x, t) - (U, V)(x - ct)| \to 0 \quad \text{as} \quad t \to +\infty.$$

(iii)

$$|(U^\varepsilon, V^\varepsilon)(z) - (U^0, V^0)(z)| \to 0, \ \text{as}\ \varepsilon \to 0.$$

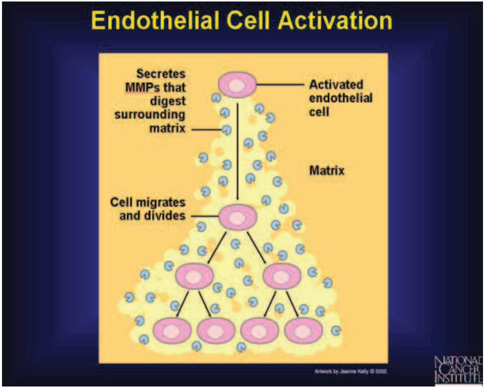
Open questions

1. Removal of smallness assumption of ε for nonlinear stability with $u_+ > 0$;
2. Stability for $u_+ = 0$ (no boundary layer);
3. Traveling waves (existence and stability) in multi-dimensions for both $\varepsilon = 0$ and $\varepsilon > 0$.

Traveling waves with cell growth

$$\begin{cases} u_t = (Du_x - \chi u v^{-1} v_x)_x + f(u) \\ v_t = \varepsilon v_{xx} - uv \end{cases}$$

In tumor angiogenesis, when endothelia cells move toward VEGF, they also divide.



Cont'd

Ai, Huang and Wang (ongoing, 2012): The variable transformation

$$w = -\frac{v_x}{v}$$

transforms the system into

$$\begin{cases} u_t - \chi(uw)_x = Du_{xx} + f(U), \\ w_t + (\varepsilon w^2 - u)_x = \varepsilon w_{xx}. \end{cases}$$

Then the wave profile $(U, W)(z) = (u, v)(x - ct)$ satisfies

$$\begin{cases} (U' + \chi UW - cU)' = -f(U), \\ -cW' = \varepsilon W'' - (\varepsilon W^2 - U)' \end{cases}$$

Define a new variable

$$V = U' + \chi UW + cU.$$

Then

$$\begin{cases} U' = V - \chi UW - cU, \\ V' = -f(U), \\ W' = W^2 - \frac{c}{\varepsilon} W - \frac{1}{\varepsilon} U. \end{cases}$$

Cont'd

Consider $f(u) = u(1-u)$ and

$$\begin{cases} U' = V - \chi UW - cU, \\ V' = -u(1-u), \\ W' = W^2 - \frac{c}{\varepsilon}W - \frac{1}{\varepsilon}U. \end{cases} \tag{9}$$

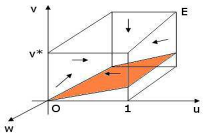
There are equilibria

$$E_c = (1, v_c^*, w_c^*), E_0 = (0,0,0)$$

where

$$w_c^* = \frac{-2}{c + \sqrt{c^2 + 4\varepsilon}}, \ \ v_c^* = \chi w_c^* + c > 0.$$

$$\mathcal{B}_c := \{(u,v,w) : 0 \le u \le 1, 0 \le v \le v_c^*, w_c^* \le w \le 0\}.$$



Theorem. For any $c > 2 + \frac{\chi}{1+\sqrt{1+\varepsilon+\chi}}$, there is a heteroclinic solution of (9) that lies in the interior of $\mathcal{B}_c$ with $U' < 0$, $V' < 0$ and $W' > 0$ on $(-\infty, \infty)$.

Works to be done

1. Finding minimum speed c_0 (confirmed)
2. Stability

Initial boundary value problem

- non-diffusive chemical ($\varepsilon = 0$)

1. 1-D case:

$$\begin{cases} u_t - (uw)_x = u_{xx}, \ \ x \in (0,1), t > 0 \\ w_t - u_x = 0 \\ (u, w)(x,t) = (u_0, w_0)(x), \ \ x \in [0,1] \\ u_x|_{x=0,1} = w|_{x=0,1} = 0, \ \ t > 0 \end{cases}$$

   - Zhang and Zhu (2007)(Proc. AMS, 135: 1017-1027): global existence for small data: if $u_0, w_0 \in H^2(0,1)$ and small, then $u, w \in\in L^\infty((0,1); H^2(0,1))$.
   - Li, Pan and Zhao (2011) (SIAM J. Appl. Math): large data and asymptotic behavior: If $u_0, w_0 \in H^2(0,1)$, then

$$\|(u - \bar{u}, w)\|_{L^\infty} \to 0 \ \ \mathrm{as} t \to \infty$$

   exponentially, where $\bar{u} = \int_0^1 u_0(x)dx$.

2. Multi-D, $\Omega \subset \mathbb{R}^N, N \geq 2$, Similar results to 1-D obtained by Li, Pan and Zhao (2011) but for small data. The results for large data remain open!

IBVP of repulsive chemotaxis model with log sensitivity

- Diffusive chemical ($\varepsilon > 0$)

1. 1-D case:

$$\begin{cases} u_t - (uw)_x = u_{xx}, \; x \in (0,1), t > 0 \\ w_t - (\varepsilon w^2 + u)_x = \varepsilon u_{xx} \\ (u,w)(x,t) = (u_0, w_0)(x), \; x \in [0,1] \\ u_x|_{x=0,1} = w|_{x=0,1} = 0, \; t > 0 \end{cases}$$

   - Wang and Zhao (2012, submitted): asymptotic behavior for large data and small ε: if $u_0, w_0 \in H^2(0,1)$ and $\varepsilon > 0$ is small, then

$$\|(u - \bar{u}, w)\|_{L^\infty} \to 0 \text{ as } \; t \to \infty$$

     exponentially, where $\bar{u} = \int_0^1 u_0(x)dx$.
   - Tao and Wang (2012, submitted): asymptotic behavior for large data with any $\varepsilon > 0$: same results as above. Furthermore it is shown only constant stationary solutions exists.

2. Multi-D: completely open!

Cauchy problem for repulsive chemotaxis model

$$\begin{cases} u_t - \nabla \cdot (u\mathbf{w}) = \Delta u, \; x \in \mathbb{R}^N, t > 0 \\ \mathbf{w}_t - \nabla(\varepsilon \mathbf{w}^2 + u) = \varepsilon \Delta w \end{cases}$$

1. Non-diffusive chemical ($\varepsilon = 0$)
   - $N = 1$: Guo, Xiao, Zhao and Zhu (2009) (Acta. Math, Sci., 29:629-641): Global existence for large data.
   - $N \geq 2$: D. Li, T. Li and K. Zhao (2011)(Math. Models. Methods. Appl. Sci., 21:1631-165): global existence for small data.
2. Diffusive chemical ($\varepsilon > 0$)
   - $N = 1$: Peng, Ruan and Zhu (2011): Global existence for strictly positive large date
   - $N = 1$: Li, Wang and Zhao (2012): large time behavior with decay rates for any non-negative initial data
   - $N \geq 2$: completely open!

Attractive RRW model with log sensitivity

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \frac{\nabla v}{v}), x \in \Omega \subset \mathbb{R}^N, t > 0 \\ v_t = \varepsilon \Delta v + uv^{\gamma} - v, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \ x \in \partial\Omega \end{cases}$$

with $\chi > 0$ and $\gamma = 0, 1$ and ν denoted the outward normal vector.

- Diffusive chemical ($\varepsilon > 0$)
    - Linear growth $\gamma = 0$:
      (1) $N \geq 1$: (least-energy) stationary solutions (C.S. Lin, W.M. Ni and I. Takagi (1988, JDE), W.M. Ni (1998, N. AMS))
      (2) $N = 1$: global classical solutions exist (K. Osaki and A. Yagi 2001)
      (3) $N \geq 2$: (P. Biler 1999, T. Nagai, T. Senba and K. Yoshida 1997 (Parabolic-elliptic)): Global classical solutions exist for $n = 2, \chi \leq 1$; M. Winkler (2010): for $n \geq 2$, global classical solutions exist if $\chi < \sqrt{2/n}$, and global weak solutions exist for $\chi < \sqrt{(n+2)/(3n-4)}$; M. Winkler (2011): $n \geq 2$, for any $\chi > 0$, global weak power-λ radial solutions exist;
      open: Whether there is L^{∞}-blowup for large χ.

Cont'd

- Exponential growth $\gamma = 1$:

  Tao and Wang (2012): $N = 1$, global classical solutions exist

  $N \geq 2$: largely open

Attractive RRW model with log sensitivity

Non-diffusive chemical ($\varepsilon = 0$)

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \frac{\nabla v}{v}), x \in \Omega \subset \mathbb{R}^N, t > 0 \\ v_t = uv^\gamma - \beta v, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \ x \in \partial\Omega \end{cases}$$

with $\beta > 0$ with $\chi = 1$.

1. Othmer and Stevens (1997): $N = 1$, the constant solution $(u_c, u_c/\beta)$ is asymptotically stable for $\gamma = 0$, and numerically show that the solution may blow up for $\gamma = 1$.
2. Levine and Sleeman (1997): solution may blow up in 1-D by explicitly constructing solutions
3. Yang, Chen and Liu (2001): $N \geq 1$, for $\gamma = 0$ unique bounded global solution exists which might collapse depending on the initial; for $\gamma = 1$, both global and finite-time blow up solutions may exist depending on the initial data by explicitly constructing exact solutions
4. T. Nagai (2005): constructing blow up solution use different approach, and large time behavior under additional conditions for initial data

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