Asymptotic behavior of nonlocal equations

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Salome Martinez
University of Chile, Chile. samartinez@dim.uchile.cl

The Center receives funding from

Cente for PDE
500 Dongchuan Road
Administration Building 12th floor
Minhang Campus, East China Normal University
Shanghai, 200241, China
Email: admin@cpde.ecnu.edu.cn

05/16. Martinez.

Types of equations

$$\begin{cases} \frac{du}{dt} = \int_{\mathbb{R}^N} K(x,y)\, u(y,t)\, dy - \int_{\mathbb{R}^N} K(y,x)\, u(x,t)\, dy \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}^N) \end{cases}$$

$u(x,t)$: density of a population at site x, time t.

$K(x,y)$: rate at which individual of site y moving to x.

$$\int_{\mathbb{R}^N} K(y,x)\, dy = 1.$$

<u>Simple cases</u>. $K(x,y) = J(x-y)$, J: radially symm.

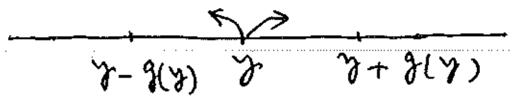
We will study the following cases:

$$K(x,y) = J\left(\frac{|x-y|}{g(y)}\right) g^{-N}(y),$$

where J: even function in $\mathbb{R}$ & nonnegative.

$g: \mathbb{R}^N \to \mathbb{R}$ continuous & positive.

Example: $\operatorname{supp} J \subset [-1, 1]$.



$$\int J\left(\frac{|x-y|}{g(x)}\right) g^{-N}(x)\, dy = 1.$$

For fixed y, $J\left(\frac{|x-y|}{g(y)}\right) g^{-N}(y)$ is supported in $B(y, g(y))$.

// We are going to study

1) $$\frac{du}{dt} = \int_{\mathbb{R}^N} J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g^N(y)}\, dy - u(x,t)$$

$N=1$

1) Asymptotic behavior of the sol.

2). Existence of positive steady state, i.e.,

$$p(x) = \int J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)}\, dy$$

3) Nonlinear problem & eigenvalue problem.

$$\frac{du}{dt} = \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\, dy - u(x,t) + (a(x) - u)\, u.$$

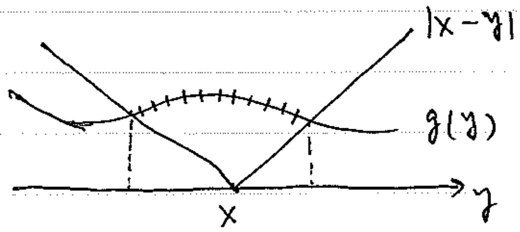
Existence of s.s. is determined by principal eigenvalue.

We make <u>some assumptions</u>

- J even supp $J = [-1, 1]$, Hölder continuous.
- $g > 0$ continuous.

$$J\left(\frac{|x-y|}{g(y)}\right) > 0 \iff |x-y| < g(y)$$

①



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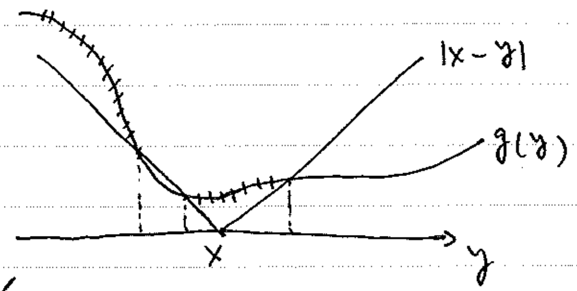
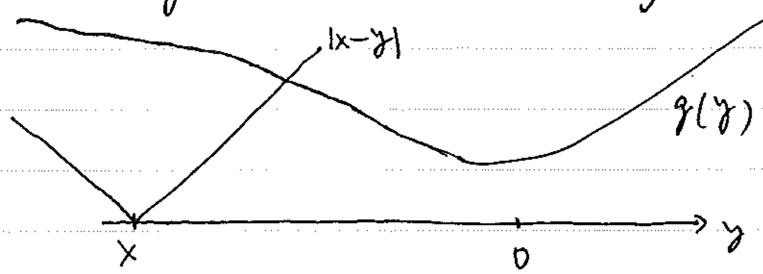


③ If $g(y)$ is large when y large, then always get contribution from large region when $-x$ large.



Today, focus on the case $\boxed{a < g < b}$

<u>Case $g \equiv 1$</u>

$$\begin{cases} u_t = \int J(x-y)\, u(y,t)\, dy - u(x,t) \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}) \end{cases}$$

Fourier transform

$$\begin{cases} \hat{u}_t = \hat{J}(\xi)\, \hat{u}(\xi,t) - \hat{u}(\xi,t) \\ \hat{u}(\xi,0) = \hat{u}_0(\xi) \end{cases}$$

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\, \hat{u}_0(\xi).$$

$\hat{J}(\xi)$:

- $\hat{J}(0) = 1$ since $\int J(y)\, dy = 1$.
- $\hat{J}$ is real (since J is even) & $\hat{J}(\xi)$ is even.
- $|\hat{J}(\xi)| < 1$, $\xi \neq 0$.
- $\hat{J}(\xi) \to 0$ as $|\xi| \to \infty$.

$$\Rightarrow \quad \hat{J}(\xi) - 1 \sim -A|\xi|^2, \quad \xi \approx 0.$$
$$\hat{u}(\xi, t) \sim e^{-A|\xi|^2 t}\, \hat{u}_0(\xi).$$

Let's consider $\begin{cases} v_t = A\Delta v \\ v(x,0) = u_0 \end{cases}$ $\left(\begin{array}{l} \hat{v}_t = -A|\xi|^2 \hat{v} \\ \hat{v}(\xi, 0) = \hat{u}_0 \\ \hat{v}(\xi, t) = e^{-A|\xi|^2 t}\, \hat{u}_0 \end{array} \right)$

Proposition $\hat{u}_0$, $u_0 \in L^1$.

$$t^{1/2} \max_{x \in \mathbb{R}} |u(x,t) - v(x,t)| \to 0 \quad \text{as} \quad t \to \infty.$$

where $u(x,t)$: $\begin{cases} u_t = J * u - u \\ u(x,0) = u_0 \end{cases}$ $\quad v(x,t)$: $\begin{cases} v_t = A\Delta v \\ v(x,0) = u_0. \end{cases}$

[Chasseigne, Chavez, Rosi).

If we try Fourier transform for our problem, ~~then~~

$$u_t = \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\, dy - u(x,t)$$

$$\hat{u}_t = \iint J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\, dy\, e^{-i\xi x}\, dx - \hat{u}$$

$$\left(z = \frac{x-y}{g(y)} \qquad dx = g(y)\, dz \right)$$

$$= \iint J(z)\, e^{-i\xi(z g(y) + y)}\, dz\, u(y,t)\, dy - \hat{u}$$

$$= \int \hat{J}(\xi g(y))\, e^{-i\xi y}\, u(y,t)\, dy - \hat{u}$$

// Suppose $a \le g \le b$.

$$\begin{cases} u_t = \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\, dy - u(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

Goal : show $u(x,t) \to 0$ in L^1_{loc}.

Basic facts:

$$Tu = \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\, dy$$

$$T: L^1 \longrightarrow L^1$$

$$\int Tu = \iint J\left(\frac{x-y}{g(y)}\right) \frac{u(y,t)}{g(y)}\, dy\, dx$$

$$= \int \underbrace{\int J\left(\frac{x-y}{g(y)}\right) \frac{1}{g(y)}\, dx}_{1}\; u(y,t)\, dy$$

$$= \int u(y,t)\, dy.$$

$\exists !$ sol. $u \in C^1(\mathbb{R}, L^1)$.

$$u_t = Tu - u$$

$$\frac{d}{dt} e^t u = e^t Tu$$

$$e^t u(x,t) - u(x,0) = \int_0^t e^s \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y,s)}{g(y)}\, dy\, ds$$

$$u(x,t) = e^{-t} u(x,0) + \int_0^t e^{s-t} \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y,s)}{g(y)}\, dy\, ds.$$

- $u(x,t)$ not smooth if $u(x,0)$ not smooth.
- Infinite speed of propagation: if $u_0(x) \geq 0$, $u_0 \not\equiv 0$, then $u(x,t) > 0$ $\forall\ t > 0$, $x \in \mathbb{R}$.
- $\int u(x,t)\, dx = \int u_0(x)\, dx$
- $\int |u(x,t)|\, dx \leq \int |u_0(x)|\, dx$

$$\int |u(x,t)|\, dx \leq e^{-t} \int |u_0(x)|\, dx + \iint_0^t e^{s-t} \int J\left(\frac{x-y}{g(y)}\right) \frac{|u(y,s)|}{g(y)}\, dy\, ds\, dx$$

$$= e^{-t} \int |u_0(x)|\, dx + \int_0^t e^{s-t} \int |u(y,s)|\, dy\, ds$$

$$\leq e^{-t} \int |u_0(x)|\, dx + \max_{s \in [0,t]} \|u(\cdot,s)\|_{L^1} \left(1 - e^{-t}\right)$$

If $\exists\ t > 0$ s.t. $\|u(\cdot,t)\|_{L^1} = \max_{s\in[0,t]} \|u(\cdot,s)\|_{L^1} > \|u_0(\cdot)\|_{L^1}$

then $e^{-t} \int |u(x,t)|\, dx \leq e^{-t} \int |u_0(x)|\, dx$

$$\Rightarrow \int |u(x,t)|\, dx \leq \int |u_0(x)|\, dx.$$ Contradiction.

To prove convergence to zero in L^1_{loc}, we will use an energy.

Def. $p(x)$, (continuous, positive), is a stationary sol. if
$$p(x) = \int J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy.$$

$p(x)$, (continuous, positive), is a super sol. (sub sol.) if
$$p(x) \geq (\leq) \int J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy.$$

Suppose that $p(x)$ is a sol. or a supersol. Then define
$$E(u)(t) = \int \frac{u^2(x,t)}{p(x)}\,dx.$$

we will show if $u(x,t)$ is a (suitable) sol. of IVP, then $E(u)'(t) < 0$.

Let's compute $\frac{d}{dt}E(u)$ formally
$$\frac{d}{dt}E(u) = \int \frac{2u\,u_t}{p(x)}\,dx$$
$$= \int \frac{2u(x,t)}{p(x)}\left[\int J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy - u(x,t)\right]dx$$

since $p(x)$ is a supersol.
$$\frac{1}{p(x)}\int J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy \leq 1.$$
$$\int \frac{u^2(x,t)}{p(x)}\,dx \geq \iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy\,\frac{u^2(x,t)}{p^2(x)}\,dx$$
$$\int \frac{u^2(y,t)}{p(y)}\,dy = \iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,\frac{u^2(y,t)}{p^2(y)}\,dy.$$

$$\frac{d}{dt}E(u) \leq 2\iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{u(x,t)\,u(y,t)}{p(y)\,p(x)}\,dy\,dx$$
$$-\iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{u^2(x,t)}{p^2(x)}\,dy\,dx$$
$$-\iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{u^2(y,t)}{p^2(y)}\,dy\,dx$$
$$= -\iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{u(x,t)}{p(x)}-\frac{u(y,t)}{p(y)}\right]^2 dy\,dx$$

roughly speaking, $\frac{u(x,t)}{p(x)} \to \lambda$ if $E(u)'(t) = 0$.

$$u(x,t) \longrightarrow \lambda\, p(x).$$

if $p \notin L^1 \Rightarrow \lambda = 0$ since $u \in L^1$.

if $p \in L^1$, then it must be a sol. & unique !!

Proposition. Suppose that $p > 0$ is a supersol., $p \in L^1$. Then p is a positive sol. & unique up to multi. by a positive constant.

Proof.
$$p(x) \geq \int J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy$$
$$\int p(x)\,dx \geq \iint J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy\,dx = \int p(y)\,dy.$$

$\Rightarrow p(x)$ is a sol.

Uniqueness. Suppose that $p(x) \in L^1$, $p_1(x)$ are two positive sol., then $w \triangleq \min\{p_1, p\}$ is a supersol. ✓.

$\Rightarrow w \in L^1$ since $p \in L^1$

$\Rightarrow w$ is a solution.

Take p_1 (multi. by a suitable constant) so that
$$p(x_0) = p_1(x_0).$$

Then $\{p(x) = p_1(x)\}$ is not empty & closed.

Remains to it is open.

If $p(x) = p_1(x)$.

$$p(x) = \int J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy \geq \int J\left(\frac{x-y}{g(y)}\right)\frac{w(y)}{g(y)}\,dy = w(x) = p(x).$$

$\Rightarrow$ $p(y) = w(y)$ in a neigh. of x since $p(y) \geq w(y)$.

Similarly, $p_1(y) = w(y)$ in a neigh. of x.

So, $\{p_1(x) = p(x)\}$ is open.

$\Rightarrow$ $p_1(x) \equiv p(x)$ ☒

Prop. If u is a continuous sol. in L^1, then u is positive.

Proof.

$$u_+(x) = \int J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\,dy$$

$$\leq \int J\left(\frac{x-y}{g(y)}\right)\frac{u_+(y)}{g(y)}\,dy.$$

$\Omega := \{x \mid u > 0\}$,

$$\int_\Omega u_+(x)\,dx \leq \int_\Omega\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u_+(y)}{g(y)}\,dy\,dx$$

$$= \int_{\mathbb{R}}\int_\Omega J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}\,dx\;u_+(y)\,dy$$

$$\leq \int_{\mathbb{R}} u_+(y)\,dy = \int_\Omega u_+(y)\,dy$$

$\Rightarrow$ $\Omega = \mathbb{R}$. ☒

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Solutions when $g \equiv 1$.

$$\int J(x-y)\,u(y)\,dy - u(x) = 0 \quad \text{in } \mathbb{R}.$$

$$\hat{J}(\xi)\,\hat{u} - \hat{u} = 0.$$

$$\hat{J}(\xi) - 1 \neq 0 \quad \text{unless } \xi = 0.$$

$\Rightarrow$ $\operatorname{supp} \hat{u}(\xi) = \{0\}$ $\Rightarrow$ $\hat{u}(\xi) = \sum C_\alpha \partial^\alpha \delta$

$\Rightarrow$ u is a polynomial.

Check $u = 1$ is a sol.
$u = x$ is also a sol.
($\int J(x-y)\, y\, dy = \int J(z)\,(z+x)\, dz = x$).
Later, can prove 1 & x are the tempered sol.

Depending on J, there are sols. which oscillate & grow exponentially.

When $a \le g(x) \le b$, construct sol. $p(x)$ sat. $d_1 \le p(x) \le d_2$.

$$T u = \int J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g(y)}\, dy.$$

we will work with a truncated problem.

$$\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g(y)}\, dy - u(x) \underbrace{\int_{-K}^{K} J\left(\frac{x-y}{g(x)}\right) \frac{1}{g(x)}\, dy}_{H_K(x).} = 0$$

$$\underbrace{\frac{1}{H_K(x)} \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right) \frac{u(y)}{g(y)}\, dy}_{L_K(u)\ :\ \text{compact in } C([-K,K]).} = u(x).$$

L_K^n is strongly positive, by Krein - Rutman Thm.,
$\exists\ \lambda > 0$, principal eigenv., $p_K(x) > 0$ corresponding eigenf.

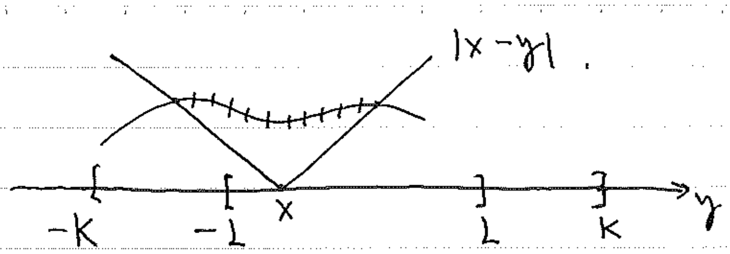
$$\frac{1}{H_K(x)} \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right) \frac{p_K(y)}{g(y)}\, dy = \lambda\, p_K(x).$$

$$\int_{-K}^{K}\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right) \frac{p_K(y)}{g(y)}\, dy\, dx = \lambda \int_{-K}^{K}\int_{-K}^{K} J\left(\frac{x-y}{g(x)}\right) \frac{p_K(x)}{g(x)}\, dx\, dy$$

$$\Rightarrow \lambda = 1.$$

<u>Need bounds on</u> $p_K(x)$.
Assume $p_K(0) = 1$.

$$\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right) \frac{p_K(y)}{g(y)}\, dy = \int_{-K}^{K} J\left(\frac{x-y}{g(x)}\right) \frac{p_K(x)}{g(x)}\, dx.$$



Fix $x \in [-L, L]$, if K is large, then

$$\int_{-\infty}^{\infty} J\left(\frac{x-y}{g(y)}\right)\frac{p_K(y)}{g(y)}\,dy - \int_{-\infty}^{\infty} J\left(\frac{x-y}{g(x)}\right)\frac{p_K(x)}{g(x)}\,dy = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} J\left(\frac{x-y}{g(y)}\right)\frac{p_K(y)}{g(y)}\,dy - p_K(x) = 0.$$

i.e. $p_K(x)$ is a sol. in $[-L, L]$.

$$p(x) = \int_{x-b}^{x+b} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy$$

$$\leq \frac{\|J\|_\infty}{a}\int_{x-b}^{x+b} p(y)\,dy.$$

$$p(0) \leq \frac{\|J\|_\infty}{a}\int_{-b}^{b} p(y)\,dy.$$

$$p(0) = \int_{-b}^{b} J\left(\frac{y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy$$

$$\geq \int_{-a/2}^{a/2} \min_{|z|\leq 1/2} J(z)\,\frac{p(y)}{b}\,dy.$$

(*) $$= \frac{1}{b}\min_{|z|\leq 1/2} J(z)\int_{-a/2}^{a/2} p(y)\,dy \triangleq \frac{1}{C_1}\int_{-a/2}^{a/2} p(y)\,dy.$$

Let's prove $\int_{-L}^{L} p(y)\,dy \leq C\,p(0)$,

where C only depends on L, J, g, not on p.

$\exists\ x_1 \in \left[-\frac{a}{2}, -\frac{a}{4}\right]$, $x_2 \in \left[\frac{a}{4}, \frac{a}{2}\right]$, s.t.

$$p(x_1) < C_1 p(0)\frac{8}{a}, \qquad p(x_2) < C_1 p(0)\frac{8}{a}.$$

If not,

$$\int_{-\frac{a}{2}}^{-a/4} p(x)\,dx > C_1 p(0)\frac{8}{a}\cdot\frac{a}{4} = 2C_1 p(0).$$ Contradicts to (*).

$$p(x) = \int_{x-b}^{x+b} J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)}\, dy$$

$$\geq \int_{x-a/2}^{x+a/2} \min_{|z|\in\frac{1}{2}} J(z) \frac{p(y)}{b}\, dy = \frac{1}{c_1}\int_{x-a/2}^{x+a/2} p(y)\, dy.$$

$$\Rightarrow \int_{x_1-a/2}^{x_1+a/2} p(y)\, dy \leq c_1 p(x_1) < c_1^2 p(0)\frac{8}{a}$$

$$\int_{x_2-a/2}^{x_2+a/2} p(y)\, dy \leq c_1 p(x_2) < c_1^2 p(0)\frac{8}{a}.$$

$$\Rightarrow \int_{-\frac{3a}{4}}^{\frac{3a}{4}} p(y)\, dy \leq c_1 p(0) + c_1^2 p(0)\frac{16}{a} \triangleq c_2 p(0)$$

Then $\exists$ $x_1' \in [-\frac{3}{4}a, -\frac{a}{2}]$, $x_2' \in [\frac{a}{2}, \frac{3}{4}a]$ s.t.

$p(x_1') < c_2 p(0)\frac{8}{a}$, $p(x_2') < c_2 p(0)\frac{8}{a}$. Repeat above arguments.

Prop. Set $[-L, L]$, L fixed, for K large

$$\int_{-L}^{L} p_K(x)\, dx \leq c(L, J, g)\, p_K(0)$$

Moreover $p_K(x) \leq c_2(L, J, g)$, $\forall\, x \in [-L, L]$.

Now, $\{p_K\}$ is bdd. in any compact set. Let's show equicontinuous.

$$|p_K(x) - p_K(z)| = \left|\int J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\, dy - \int J\left(\frac{z-y}{g(y)}\right)\frac{p(y)}{g(y)}\, dy\right|$$

$$\leq c\int \left|\frac{x-y}{g(y)} - \frac{z-y}{g(y)}\right|^{\alpha} \frac{p(y)}{g(y)}\, dy$$

$$= c\int \frac{|x-z|^{\alpha}}{g^{\alpha}(y)}\frac{p(y)}{g(y)}\, dy$$

$$\leq c\,|x-z|^{\alpha}.$$

So, up to a sub seq. $p_K \to p$, which is a positive sol. with $p(0) = 1$.