Exact solutions and the global bifurcation analysis for equations with nonlocal constraint

Lecture notes from Program on Nonlocal Effects in Differential Equations
Center for PDE, ECNU, mid February-June, 2014

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This is a tentative version of a lecture note.
Final version will appear after lectures.

1 Plan of lectures

- **Exact solutions and the global bifurcation analysis of a cell polarization model**
- **Elliptic functions and complete elliptic integrals**
- **Sturum's Theorem for the number of zeros for polynomials**
- **Inequalities including complete elliptic integrals**
- **Exact solutions and the global bifurcation analysis of one dimensional stationary Cahn-Hilliard equation**
- **Related nonlocal problems** •

In my lecture, I will mainly use PowerPoint, and show movies and 3d figures of mathematical software Maple. They help our intuitive understanding of facts and mathematical proofs.

Please check definitions and details of explanation by this tentative lecture note during and after lecture. I may version up this note even just before lectures, so check it.

The first topic "cell polarization model" is one of the topics which I am concentrate on with my PhD student Tatsuki Mori. Prof. Kuto, Prof. Tsugikawa, Prof. Nagayama, Mori. is now visiting this center with me. Most of mathematical computations by Maple, and numerical computation for time dependent problems are done by him. He will answer any question about the details of their computations.

To understand "cell polarization model", first we must obtain all exact solutions. It is not so easy, however we can do it. This problem is closely related to the topic "stationary Cahn-Hilliard equation".

For "stationary Cahn-Hilliard equation", Kosugi-Morita-Y obtained all exact solutions, and reveal the global bifurcation structure completely by using them. Thus, we need to understand them to investigate "cell polarization model"

Finally, I explain related topics which I have been studied with collaborator. Each problem has own very interesting character.

2 Cell poralization model

We are interesting in wave-pinning in a reaction-diffusion model for cell polarization proposed by Y. Mori, A. Jilkine and L. Edelstein-Keshet in SIAM J. Appl. Math (2011). The model is

$$
\text{(TP)}\begin{cases}
\varepsilon W_t = \varepsilon^2 W_{xx} + W(V+1-W)(W-1), & x \in (0,1), t \in (0,\infty),\\
\varepsilon V_t = D V_{xx} - W(V+1-W)(W-1), & x \in (0,1), t \in (0,\infty),\\
W_x(0,t) = W_x(1,t) = V_x(0,t) = V_x(1,t) = 0, & \\
W(x,0) = W_0(x),\, V(x,0) = V_0(x). &
\end{cases}
$$

Wave-pinning means a phenomenon that a wave of activation of one of the species is initiated at one end of the domain, moves into the domain, decelerates, and eventually stops inside the domain, forming a stationary front.

Several partial mathematical bifurcation results of stationary solutions

$$
\text{(SP)}\begin{cases}
\varepsilon^2 W_{xx} + W(V+1-W)(W-1) = 0, & x \in (0,1), t \in (0,\infty),\\
D V_{xx} - W(V+1-W)(W-1) = 0, & x \in (0,1), t \in (0,\infty),\\
W_x(0) = W_x(1) = V_x(0) = V_x(1) = 0, & \\
\int_0^1 (W(x)+V(x))\,dx = \int_0^1 (W_0(x)+V_0(x))\,dx = m &
\end{cases}
$$

and a stationary limiting equation as $D \to \infty$

$$
(\text{SP}\,\infty)\begin{cases}
\varepsilon^2 W_{xx} + W(\tilde{V}+1-W)(W-1) = 0, & x \in (0,1),\\
W_x(0) = W_x(1) = 0, & \\
\int_0^1 W dx + \tilde{V} = m &
\end{cases}
$$

are obtained by Kuto and Tsujikawa in DCDS Supplement (2013).

We propose a new method to represent a bifurcation sheet of a limiting system. It determines the global bifurcation structure of stationary solutions of the limiting system completely including even secondary bifurcation branches. Moreover, we numerically investigate the global bifurcation structure and stability of the original reaction-diffusion model to understand the wave-pinning.

For simplicity we concentrate on monotone increasing solutions.

$$
\text{(SLP)}\begin{cases}
\varepsilon^2 W_{xx} + W(\tilde{V}+1-W)(W-1) = 0, \qquad x \in (0,1), & (1)\\
W_x(0) = W_x(1) = 0, & (2)\\
W_x(x) > 0 & (3)\\
\displaystyle\int_0^1 W dx + \tilde{V} = m & (4)
\end{cases}
$$

First, we consider the following auxiliary problem (1), (2), (3). We have the following results.

Theorem 2.1..

(i) *All solutions of boundary value problem (1), (2), (3) are equivalent to the following system of transcendental equations for* $(h,\ s)$

$$
\text{(E)}\begin{cases}
\mathcal{E}(h,s) = \dfrac{\varepsilon}{\tilde{V}+1}\Big/\sqrt{\dfrac{1}{3}\left((\dfrac{1}{\tilde{V}+1}-\dfrac{1}{2})^2+\dfrac{3}{4}\right)},\\
\mathcal{A}(h,s) = -\dfrac{2}{27}\left((\dfrac{1}{\tilde{V}+1}-\dfrac{1}{2})^3-\dfrac{9}{4}(\dfrac{1}{\tilde{V}+1}-\dfrac{1}{2})\right)\\
\qquad\qquad \Big/\sqrt{\dfrac{1}{3}\left((\dfrac{1}{\tilde{V}+1}-\dfrac{1}{2})^2+\dfrac{3}{4}\right)}^{\,3},
\end{cases}
$$

where

$$
\mathcal{E}(h,s) := \frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(1+h)s+3}},
$$

$$
\mathcal{A}(h,s) := \frac{2(hs^2-2sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^{\,3}}.
$$

(ii) *There exists the unique solutions of (E), if and only if* $(\tilde{V}, \varepsilon) \in \mathcal{G}$, *where,* $\mathcal{G} := \left\{ (V, \varepsilon) : 0 < V,\ 0 < \varepsilon < \dfrac{\sqrt{V}}{\pi} \right\}$.

Moreover, let $(h,\ s) = (h(\tilde{V}, \varepsilon),\ s(\tilde{V}, \varepsilon))$*, be the unique solutions then,*

$$\begin{aligned} W(x, \tilde{V}, h, s) &= \frac{\tilde{V}+2}{3} \\ &+ (\tilde{V}+1)\sqrt{\frac{1}{3}\left((\frac{1}{\tilde{V}+1} - \frac{1}{2})^2 + \frac{3}{4} \right)} \\ &\cdot \frac{\beta \cdot (1-hs)\ \mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha \cdot \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)\ \mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}, \end{aligned}$$

$$\alpha := \alpha(h, s) = \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}},$$

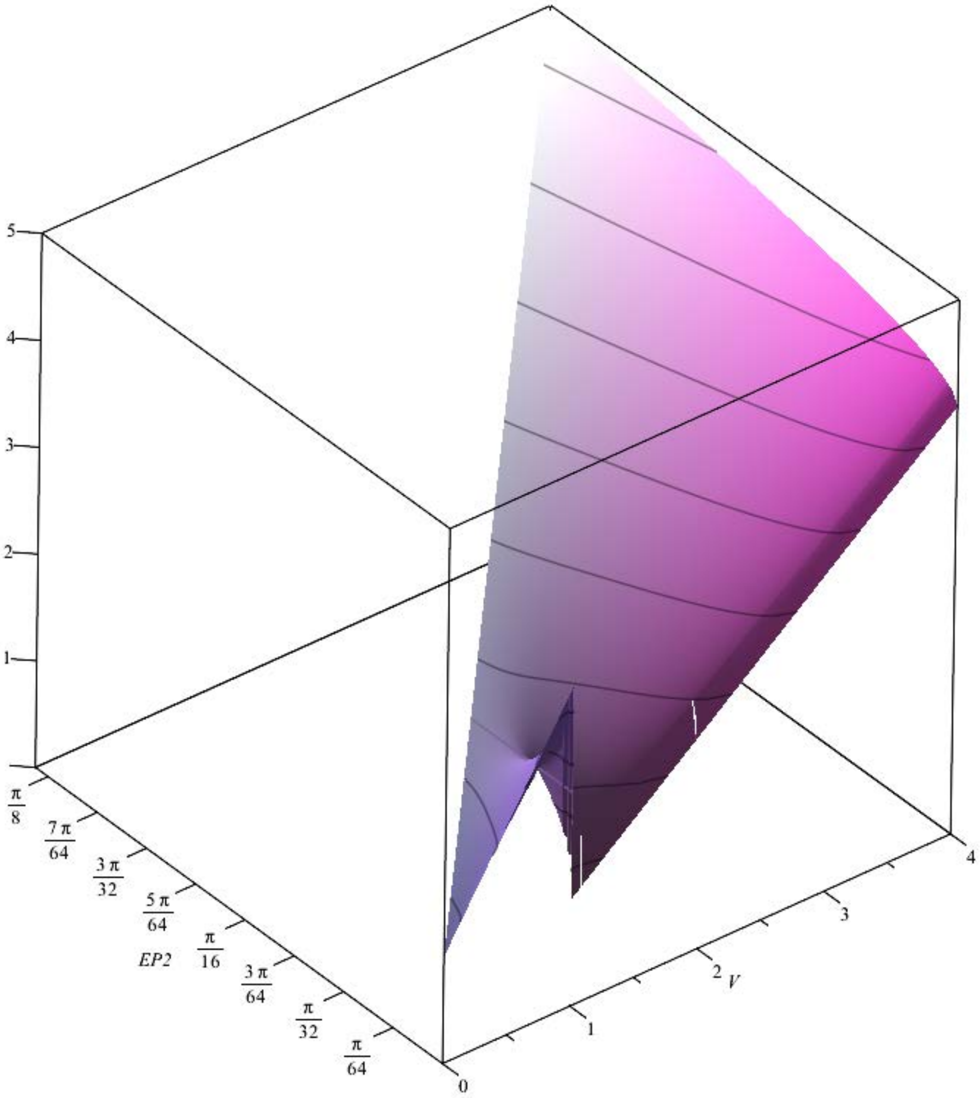
$$\beta := \beta(h, s) = \frac{-hs^2 - 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}.$$

(iii) *Global bifurcation sheet for solutions of (SLP) is represented by*

$$\left\{ \tilde{V}, \varepsilon, \frac{\dfrac{1}{\tilde{V}+1}}{I(\tilde{V}, h(\tilde{V}, \varepsilon),\ s(\tilde{V}, \varepsilon))} : (\tilde{V}, \varepsilon) \in \mathcal{G} \right\},$$

where,

$$I(V, h, s) := \frac{4V+2}{3} + \sqrt{\frac{1}{3}\left((\frac{1}{\tilde{V}+1} - \frac{1}{2})^2 + \frac{3}{4} \right)} \cdot \mathcal{M}(h, s)$$



For given each m, the bifurcation diagram is a level set of hight m of this bifurcation *sheet*. We need to investigate the various properties of the bifurcation.sheet. I will explain the detail of them in my lectures.

I also show numerical results of the stability of solutions of the limiting solutions and the original equation.

3 Elliptic functions

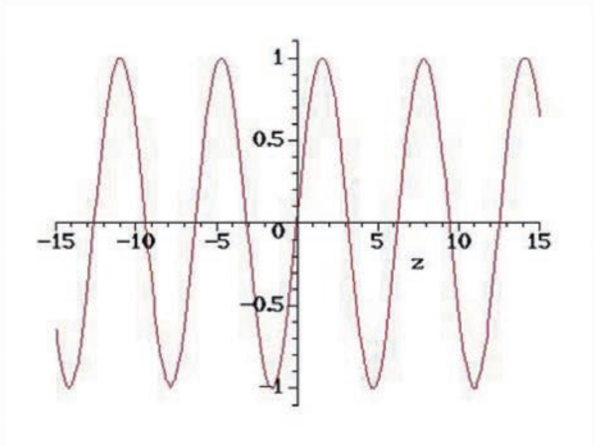
Jacobi's elliptic functions $\mathrm{sn}(x, k)$ with the modulus k are defined as follows:

$$\mathrm{sn}^{-1}(z, k) := \int_0^z \frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \quad z \in [0,1],\ k \in [0,1).$$

The function $\mathrm{sn}(x, k)$ is defined on $(-\infty, \infty)$ by extending the original piece on $[0, K(k)]$ to $(-\infty, \infty)$ like $\sin(x)$. Here $K(k)$ is the complete elliptic integral of the first kind

$$K(k) := \int_0^1 \frac{1}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}\, d\xi.$$

The period of $\mathrm{sn}(x, k)$ is $4K(k)$, and it moves from 2π to ∞ as k moves from 0 to ∞. The graph is $\mathrm{sn}(x, k)$ as follows:



We note that

$$\mathrm{sn}(x, 0) = \sin(x), \quad K(0) = \frac{\pi}{2}$$

$K(k)$ is increasing in $k \in [0, 1)$, $\quad K(k) \to \infty$ as $k \to 1$.

The Jacob's elliptic function $\mathrm{cn}(x, k)$ is defined like from $\sin(x)$ to $\cos(x)$, thus the following identity holds:

$$\mathrm{cn}^2(x, k) = 1 - \mathrm{sn}^2(x, k).$$

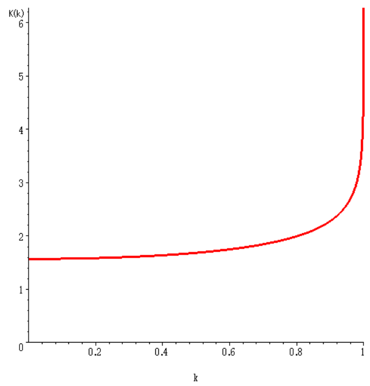
We note that $\mathrm{cn}(x, 0) = \cos(x)$.

4 Complete elliptic integrals

the first kind complete elliptic integral

$$K(k) := \int_0^1 \frac{1}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}\, d\xi = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi,$$

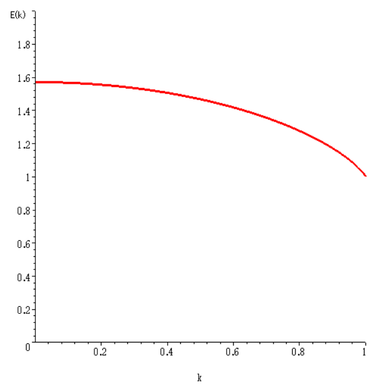
$K(0) = \pi/2,$ $\qquad K(k) \to \infty$ as $k \to 1$



the second kind complete elliptic integral

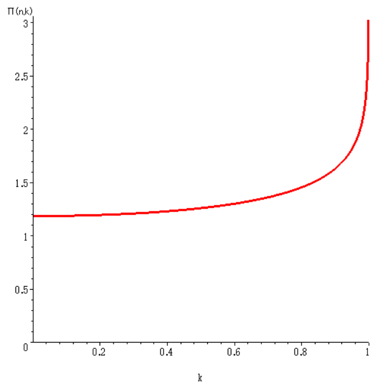
$$E(k) := \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\, d\xi = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi,$$

$E(0) = \pi/2,$ $\qquad E(1) = 1$



the third kind complete elliptic integral

$$\Pi(n,k) = \int_0^1 \frac{1}{(1+n\xi^2)\sqrt{(1-\xi^2)(1-k^2\xi^2)}} d\xi$$



fundamental formula

$$\frac{d}{dk}K(k) = \frac{E(k)}{(1-k^2)k} - \frac{K(k)}{k},$$

$$\frac{d}{dk}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}$$

$$\frac{\partial}{\partial k}\Pi(n,k) = \frac{kE(k)}{(k^2+n)(1-k^2)} - \frac{k\Pi(n,k)}{(k^2+n)},$$

$$\frac{\partial}{\partial n}\Pi(n,k) = \frac{(k^2-n^2)\Pi(n,k)}{2(1+n)(k^2+n)n} - \frac{K(k)}{2(1+n)n}p + \frac{E(k)}{2(1+n)(k^2+n)}$$

Proof of fundamental formula (only for easy ones)

$$K(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

$$E(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,d\varphi$$

$$\begin{aligned}
&\frac{dE(K)}{dk} = \frac{d}{dk}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta \\
=\ & \int_0^{\pi/2}\frac{\partial}{\partial k}\sqrt{1-k^2\sin^2\theta}d\theta = \int_0^{\pi/2}\frac{-k\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}} \\
=\ & \frac{1}{k}\left(\int_0^{\pi/2}\frac{1-k^2\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta - \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\right) \\
=\ & \frac{1}{k}(E(k)-K(k))
\end{aligned}$$

$$\begin{aligned}
&\frac{dK(K)}{dk} = \frac{d}{dk}\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta = \int_0^{\pi/2}\frac{\partial}{\partial k}\left(\frac{1}{\sqrt{1-k^2\sin^2\theta}}\right)d\theta \\
=\ & \int_0^{\pi/2}\frac{k\sin^2\theta}{(1-k^2\sin^2\theta)^{3/2}}d\theta = \int_0^{\pi/2}\frac{k}{1-k^2}\frac{d}{d\theta}\left(-\frac{\cos\theta}{\sqrt{1-k^2\sin^2\theta}}\right)\sin\theta d\theta \\
=\ & \frac{k}{1-k^2}\left[-\frac{\cos\theta\sin\theta}{\sqrt{1-k^2\sin^2\theta}}\right]_0^{\pi/2} + \frac{k}{1-k^2}\int_0^{\pi/2}\frac{\cos^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta \\
=\ & \frac{k}{1-k^2}\int_0^{\pi/2}\frac{1}{k^2}\frac{1-k^2\sin^2\theta-(1-k^2)}{\sqrt{1-k^2\sin^2\theta}}d\theta \\
=\ & \frac{1}{k(1-k^2)}\left(\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta - (1-k^2)\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta\right) \\
=\ & \frac{1}{k(1-k^2)}(E(k)-(1-k^2)K(k)).
\end{aligned}$$

5 Sturm's Theorem for zeros

The Euclid Algorithm

Example. $(42, 24) = 6$

In fact,

$$42 = 1 \times 24 + 18$$

$$24 = 1 \times 18 + 6$$

$$18 = 3 \times 6.$$

Example. Let $f(x) := x^4 - 11x^3 + 44x^2 - 76x + 48..$ Then,

$$(f(x), f'(x)) = x - 2$$

In fact,

$$f(x) := x^4 - 11x^3 + 44x^2 - 76x + 48.,$$

$$f'(x) := 4x^3 - 33x^2 + 88x - 76.$$

$$f(x) = \left(\frac{x}{4} - \frac{11}{16}\right) f'(x) + \left(-\frac{(x-2)(11x-34)}{16}\right),$$

$$f'(x) = 16\left(-\frac{4x}{11} + \frac{139}{121}\right) \cdot \left(-\frac{(x-2)(11x-34)}{16}\right) + \frac{128}{121}(-x+2)$$

$$-\frac{(x-2)(11x-34)}{16} = \left(\frac{11x}{16} - \frac{17}{8}\right)(-x+2).$$

Example. Let $f(x) := x^4 - 18x^3 + 119x^2 - 342x + 360$. Then,

$$(f(x), f'(x)) = 1$$

In fact,

$f(x) := x^4 - 18x^3 + 119x^2 - 342x + 360.$ $f'(x) := 4x^3 - 54x^2 + 238x - 342$

$$f(x) = \left(\frac{x}{4} - \frac{9}{8}\right) f'(x) + \left(-\frac{5}{4}x^2 + \frac{45}{4}x - \frac{99}{4}\right),$$

$$f'(x) = \left(-\frac{16x}{5} + \frac{72}{5}\right)\left(-\frac{5}{4}x^2 + \frac{45}{4}x - \frac{99}{4}\right) + \left(-\frac{16x}{5} + \frac{72}{5}\right)$$

$$-\frac{5}{4}x^2 + \frac{45}{4}x - \frac{99}{4} = \left(\frac{25x}{64} - \frac{225}{128}\right)\left(-\frac{16x}{5} + \frac{72}{5}\right) + \frac{9}{16}$$

Example (The Euclid Algorithm)

$$f(x) := x^3 + x^2 - 3x + 1, \; f'(x) := 3x^2 + 2x - 3.$$

$$f(x) = \left(\tfrac{x}{3} + \tfrac{1}{9}\right) f'(x) + \left(-\tfrac{20}{9}x + \tfrac{4}{3}\right),$$

$$f'(x) = \left(-\frac{27}{20}x - \frac{171}{100}\right)\left(-\frac{20}{9}x + \frac{4}{3}\right) + \left(-\frac{18}{25}\right)$$

Sturm sequence

$$f(x) := x^3 + x^2 - 3x + 1, \; f'(x) := 3x^2 + 2x - 3$$

$$f(x) = \left(\frac{x}{3} + \frac{1}{9}\right) f'(x) - \left(\frac{20}{9}x - \frac{4}{3}\right),$$

$$f'(x) = \left(\frac{27}{20}x + \frac{171}{100}\right)\left(\frac{20}{9}x - \frac{4}{3}\right) - \frac{18}{25}$$

A Sturm sequence for $f(x)$ is defined by

$\{f_0(x), f_1(x), f_2(x), f_3(x)\}$,

where

$$f_0(x) = f(x), \quad f_1(x) = f'(x) = 3x^3 + 2x - 3,$$

$$f_2(x) = \frac{20}{9}x - \frac{4}{3}, \quad f_3(x) = \frac{18}{25}.$$

Define $V(x)$ by

$V(x) :=$ number of sign changing of Sturm sequence for $f(x)$.

Example $f(x) = x^3 + x^2 - 3x + 1$

$f_0(x) = f(x), \quad f_1(x) = 3x^3 + 2x - 3, \; f_2(x) = \dfrac{20}{9}x - \dfrac{4}{3}, \quad f_3(x) = \dfrac{18}{25}.$

$f_0(-2) = 3, \; f_1(-2) = 5, f_2(-2) = -\dfrac{52}{9}, \; f_3(-2) = n\dfrac{18}{25},$

$f_0(2) = 7, \quad f_1(2) = 13, \quad f_2(2) = \dfrac{28}{9}, \quad f_3(-2) = \dfrac{18}{25}.$

	f_0	f_1	f_2	f_3	V
$x = -2$	$+$	$+$	$-$	$+$	2
$x = 2$	$+$	$+$	$+$	$+$	0

$V(-2) = 2$

$V(2) = 0.$

Theorem (Strum) If $f(x) = 0$ does not have double roots and $f(a) \neq 0, f(b) \neq 0$, then the number N of real zeros in (a, b) is given by

$$N = V(a) - V(b).$$

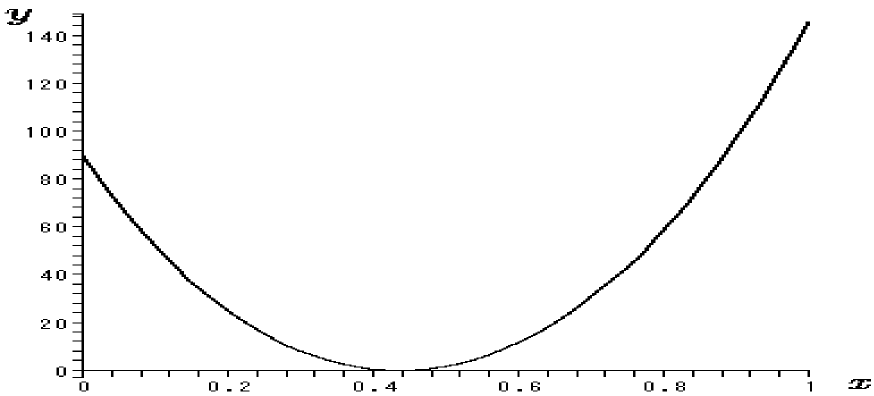
Example $x^3 + x^2 - 3x + 1 = 0 \quad in\ (-2, 2)$

$f(x) = 0$ does not have double root, $f(-2) = 3 \neq 0, f(2) = 7 \neq 0$. Therefore we obtain

$$N := V(-2) - V(2) = 2 - 0 = 2.$$

Remark $f(x) := x^3 + x^2 - 3x + 1 = (x^2 + 2x - 1).(x - 1)$
Roots of $f(x) = 0$ are $-1 - \sqrt{2},\ -1 + \sqrt{2},\ 1.$

Example $f(x) := 125x^4 - 250x^3 + 621x^2 - 440x + 90 = 0\ in\ (0, 1).$



$f'(x) = 500x^3 - 750x^2 + 1242x - 440 = 0$
$f'(0.4355\cdots) = 0?, \quad f(0.4355\cdots) = 0.006\cdots?.$

Is this argument reliable ?

$$f(x) := 125x^4 - 250x^3 + 621x^2 - 440x + 90 = 0$$
$$f'(x) = 500x^3 - 750x^2 + 1242x - 440 = 0$$

$$f_0(x) = f(x), \quad f_1(x) = f'(x)$$

$$f_2(x) = (-867\ x^2 + 699\ x - 140)/4,$$
$$f_3(x) = -(220894496x - 96212720)/250563,$$
$$f_4(x) = 4629216988985/762412161923344$$

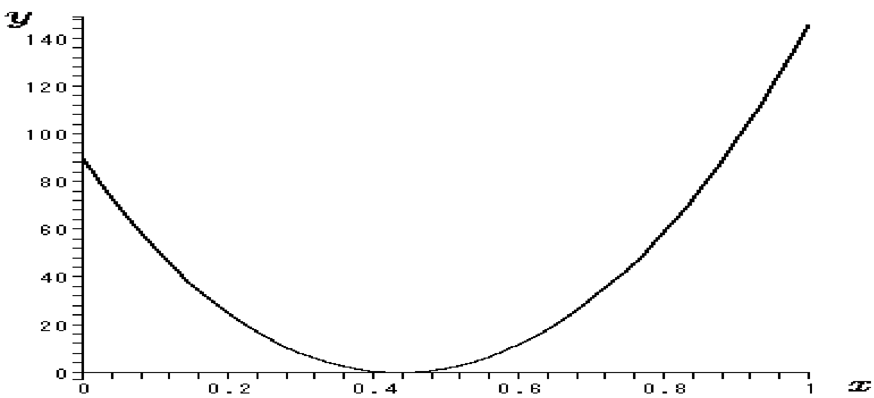
$$f_0(0) = 90, \quad f_1(0) = -440, \quad f_2(0) = -35,$$
$$f_3(0) = 96212720/250563,$$
$$f_4(0) = 4629216988985/762412161923344,$$

$$f_0(1) = 146, \quad f_1(1) = 552, \quad f_2(1) = -77,$$
$$f_3(1) = -41560592/83521,$$
$$f_4(1) = 4629216988985/762412161923344.$$

Therefore we obtain

$$V(0) - V(1) = 2 - 2 = 0.$$

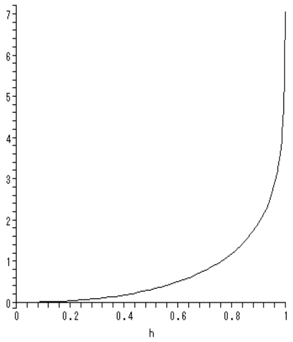
Therefore, there exists no real root in $(0, 1)$



6 Inequalities of complete elliptic integrals

Problem 6.1. *Prove the following inequality*

$$-3E(\sqrt{h})^2 + 2(2-h)K(\sqrt{h})E(\sqrt{h}) - (1-h)K(\sqrt{h})^2 > 0 \ (0 < h < 1).$$



<u>Usual answer</u>

$$f(h) := -3E(\sqrt{h})^2 + 2(2-h)K(\sqrt{h})E(\sqrt{h}) - (1-h)K(\sqrt{h})^2.$$

By $K(0) = E(0) = \pi/2$,

$$\begin{aligned} f(0) &= -3E(0)^2 + 4K(0)E(0) - K(0)^2 \\ &= -3\left(\frac{\pi}{2}\right)^2 + 4 \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} - \left(\frac{\pi}{2}\right)^2 = 0. \end{aligned}$$

By

$$\begin{aligned} \frac{d}{dk}K(k) &= \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)}, \\ \frac{d}{dk}E(k) &= \frac{E(k) - K(k)}{k}, \end{aligned} \tag{5}$$

we obtain

$$\begin{aligned} \frac{d}{dh}K(\sqrt{h}) &= \frac{1}{2} \cdot \frac{E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)}, \\ \frac{d}{dk}E(\sqrt{h}) &= \frac{1}{2} \cdot \frac{E(\sqrt{h}) - K(\sqrt{h})}{h}. \end{aligned} \tag{6}$$

Hence

$$
\begin{aligned}
\frac{d}{dh}f(h) = &-3E(\sqrt{h})\left(\frac{E(\sqrt{h})-K(\sqrt{h})}{h}\right) - 2K(\sqrt{h})E(\sqrt{h}) \\
&+2(2-h)\left(\frac{E(\sqrt{h})-(1-h)K(\sqrt{h})}{2h(1-h)}\right)E(\sqrt{h}) \\
&+2(2-h)K(\sqrt{h})\left(\frac{E(\sqrt{h})-K(\sqrt{h})}{2h}\right) \\
&+K(\sqrt{h})^2 \\
&-(1-h)K(\sqrt{h})\left(\frac{E(\sqrt{h})-(1-h)K(\sqrt{h})}{h(1-h)}\right) \\
&= \frac{\big(E(\sqrt{h})-(1-h)K(\sqrt{h})\big)}{h(1-h)}\cdot\big((1-h)K(\sqrt{h})-(1-2h)E(\sqrt{h})\big).
\end{aligned}
$$

We note that

$$
\begin{aligned}
&E(\sqrt{h})-(1-h)K(\sqrt{h}) \\
&= \int_0^{\pi/2}\sqrt{1-h\sin^2\varphi}\,d\varphi - \int_0^{\pi/2}\frac{1-h}{\sqrt{1-h\sin\varphi^2}}\,d\varphi \\
&= h\int_0^{\pi/2}\frac{\cos^2\varphi}{\sqrt{1-h\sin^2\varphi}}\,d\varphi > 0.
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-h)K(\sqrt{h})-(1-2h)E(\sqrt{h}) \\
&= (1-h)\big(K(\sqrt{h})-E(\sqrt{h})\big) + hE(\sqrt{h}) > 0.
\end{aligned}
$$

Therefore

$$
\frac{d}{dh}f(h) > 0 \quad (0 < h < 1).
$$

□

In the middle, expressions become complicated, since we differentiate a given expression. However, in this problem, we could factorized the final expression. We were so lucky.! In general, we can not expect it.

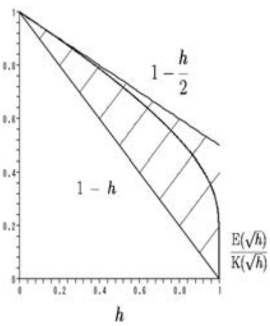
We must find a general method.

Answer (New method)

$$-3\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^2+2(2-h)\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)-(1-h)>0 \quad (0<h<1)$$

We note that the following inequality holds (easy to prove):

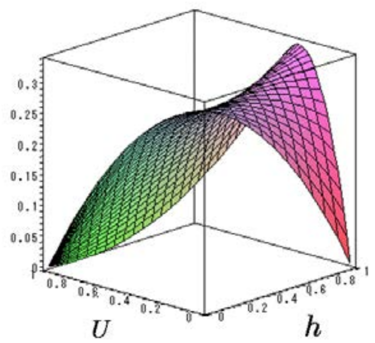
$$1-h<\frac{E(\sqrt{h})}{K(\sqrt{h})}<1-\frac{h}{2} \quad (0<h<1)$$

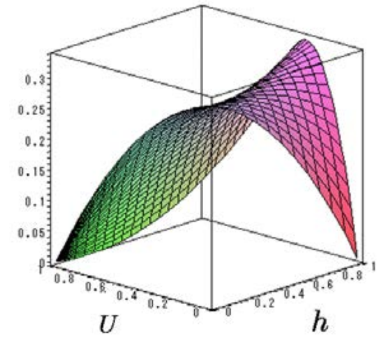


We may show that $f(h,U)>0$ on D, where

$$D:=\left\{(h,U)\,;\,0<h<1,\,1-h<U<1-\frac{h}{2}\right\},$$

$$f(h,U):=-3U^2+2(2-h)U-(1-h).$$





$$f\left(h, 1-\frac{h}{2}\right) = \frac{h^2}{4} > 0, \qquad (0 < h < 1),$$
$$f(h, 1-h) = h(1-h) > 0 \quad (0 < h < 1),$$
$$f(1, U) = U(2-3U) > 0$$

Moreover,

$$\frac{\partial f}{\partial h} = -2U + 1 = 0, \quad \frac{\partial f}{\partial U} = -6U + 2(2-h) = 0$$

implies

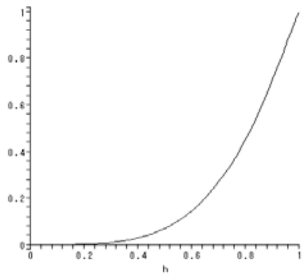
$$(h, u) = (1/2, 1/2) \notin D.$$

Consequently,

$$f(h, U) > 0 \quad in\ D.$$

Problem 6.2. *Prove the following inequality*

$$(h^2-h+1)E(\sqrt{h})^4-2(1-h)(2-h)E(\sqrt{h})^3K(\sqrt{h})$$
$$+6(1-h)^2E(\sqrt{h})^2K(\sqrt{h})^2-2(2-h)(1-h)^2E(\sqrt{h})K(\sqrt{h})^3$$
$$+(1-h)^3K(\sqrt{h})^4>0. \qquad (0<h<1)$$

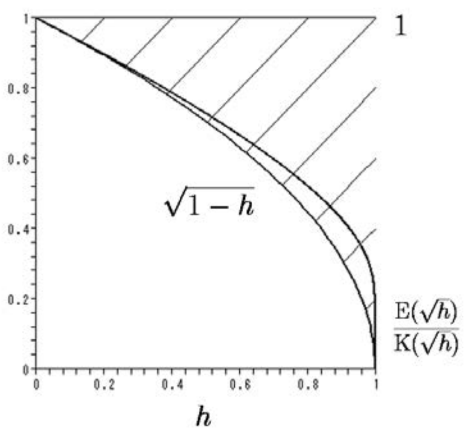


Answer We may show

$$(h^2-h+1)\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^4-2(1-h)(2-h)\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^3$$
$$+6(1-h)^2\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^2-2(2-h)(1-h)^2\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)$$
$$+(1-h)^3>0 \quad (0<h<1).$$

It hold that

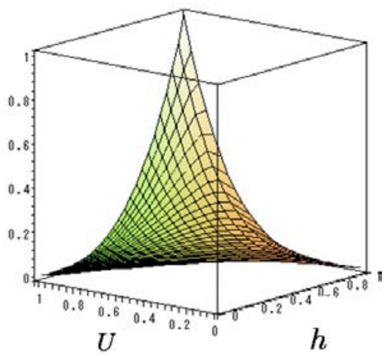
$$\sqrt{1-h}<\frac{E(\sqrt{h})}{K(\sqrt{h})}<1 \quad (0<h<1).$$



We may show in $f(h,U)>0$ in D, where

$$D := \left\{(h,U);\ 0<h<1,\ \sqrt{1-h}<U<1\right\},$$

$$f(h,U) := (h^2-h+1)U^4 - 2(1-h)(2-h)U^3 + 6(1-h)^2U^2 \\ -2(2-h)(1-h)^2U + (1-h)^3.$$



We have

$$f(h,1) = h^3 > 0 \qquad (0<h<1),$$
$$f(h,\sqrt{1-h}) = (1-h)^2(1-\sqrt{1-h})^4 > 0 \qquad (0<h<1),$$
$$f(1,U) = U^4 > 0 \quad (0<U<1).$$

Let us investigate the following system of algebraic equation

$$\begin{aligned} f_h &= (2h-1)U^4 + 2(3-2h)U^3 - 12(1-h)U^2 \\ &\quad +2(1-h)(5-3h)U - 3(1-h)^2 = 0, \\ f_U &= 4(h^2-h+1)U^3 - 6(1-h)(2-h)U^2 \\ &\quad +12(1-h)^2U - 2(2-h)(1-h)^2 = 0. \end{aligned}$$

We will show that there exists no solution in

$$\{(h,U) : 0<h<1, 0<U<1\}.$$

By using the Buchburger algorism (Groebner basis), we see that the above equation is equivalent to the following system with three equations.:

$$(1-2h)h^3(1-h)^3\cdot(125h^4-250h^3+621h^2-496h+128)=0,$$

$$h(1-h)(4250h^7-14875h^6+34989h^5-50285h^4 +34237h^3-8508h^2-192h+384-384U)=0,$$

$$864U^3-2592(1-h)U^2+2592(1-h)U +(1-h)^2\cdot(22000h^7-55000h^6+125046h^5 -132569h^4+42251h^3-864h^2-1728h-864)=0.$$

We see from the Sturm theorem for zeros that

$$125h^4-250h^3+621h^2-496h+128>0\ (0<h<1).$$

Thus $h=1/2$, which implies $U=1/2$. Cnsequently there exists no solution in D since $(h,U)=(1/2,1/2)\notin D$.

7 Comparison function for E/K and $1/K$

Theorem 7.1 ([3]). *Set* $\overline{g}_n(h)$, $\underline{g}_n(h)$ *by*

$$\begin{aligned} \overline{g}_n(h) &:= 1 - \sum_{m=0}^{n} 2^{m-1} c_m(h)^2, \\ \underline{g}_n(h) &:= 1 - \sum_{m=0}^{n} 2^{m-1} c_m(h)^2 - 2^{n-1} c_n(h)^2. \end{aligned}$$

Then

$$\underline{g}_n(h) \le \frac{E(\sqrt{h})}{K(\sqrt{h})} \le \overline{g}_n(h) \quad on\ [0,1] \quad (n = 0, 1, 2, \cdots).,$$

and

$$\begin{aligned} &\text{left} \le \text{becomes} = \iff h = 0, 1, \\ &\text{right} \le \text{becomes} = \iff h = 0, \end{aligned}$$

Moreover,

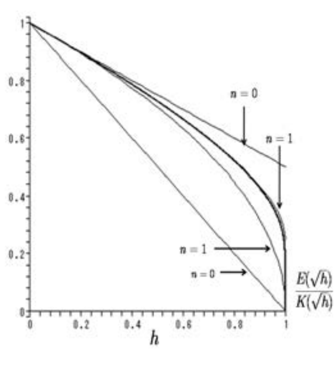
$$\underline{g}_n(h),\ \overline{g}_n(h) \to \frac{E(\sqrt{h})}{K(\sqrt{h})} \ \textit{uniformly on } [0,1].$$

Here, $c_n(h)$ are defined later. We see that

$$\begin{aligned} \overline{g}_0(h) &= 1 - \frac{h}{2}, \\ \overline{g}_1(h) &= \frac{1}{2} - \frac{h}{4} + \frac{(1-h)^{1/2}}{2}, \\ \overline{g}_2(h) &= \frac{1}{4} - \frac{h}{8} - \frac{(1-h)^{1/2}}{4} + \frac{(1-h)^{1/4}}{2} + \frac{(1-h)^{3/4}}{2}, \end{aligned}$$

$\cdots$

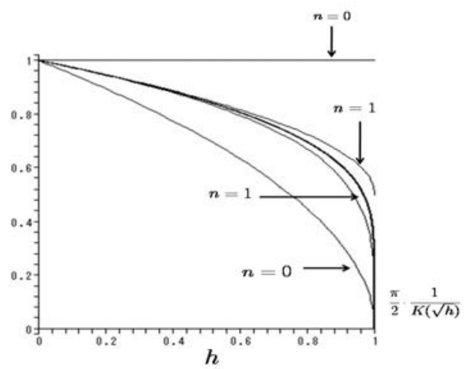
$$\begin{aligned} \underline{g}_2(h) &= (1-h)^{1/4} + (1-h)^{3/4} - (1-h)^{1/2} \\ \underline{g}_1(h) &= (1-h)^{1/2}, \\ \underline{g}_0(h) &= 1 - h. \end{aligned}$$



Theorem 7.2.

$$\frac{2}{\pi} \cdot b_n(h) \leq \frac{1}{K(\sqrt{h})} \leq \frac{2}{\pi} \cdot a_n(h) \quad (n = 0, 1, 2, \cdots).$$

$$\begin{aligned}
a_0(h) &= 1, \\
a_1(h) &= \frac{1 + (1-h)^{1/2}}{2}, \\
a_2(h) &= \frac{1 + (1-h)^{1/2}}{4} + \frac{(1-h)^{1/4}}{2}, \\
&\cdots \\
b_2(h) &= (1-h)^{1/8} \left(\frac{1 + (1-h)^{1/2}}{2} \right)^{1/2}, \\
b_1(h) &= (1-h)^{1/4}, \\
b_0(h) &= (1-h)^{1/2}.
\end{aligned}$$



Theorem 2 is by Gauss. Theorem 1 is direct application of a theorem by Gauss in 1809 concerning recursive limit of arithmetic algebra mean. We will explain the idea of proofs of them.

Let $a, b \ (a \geq b \geq 0)$ be given. Define $\{a_n\}, \ \{b_n\}$ by

$$a_0 = a, \ b_0 = b,$$
$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}. \quad (n = 0, 1, 2, \cdots)$$

and $\{c_n\}$ by

$$c_n = \sqrt{a_n^2 - b_n^2} \quad (n = 0, 1, 2, \cdots).$$

We have

$$b = b_0 \leq b_1 \leq \cdots \leq b_n \leq a_n \leq \cdots \leq a_1 \leq a_0 = a,$$

$$\frac{a_n - b_n}{2} = c_{n+1} \leq \frac{c_n}{2} \leq \cdots \leq \frac{c_0}{2^{n+1}} \quad (n = 0, 1, 2, \cdots).$$

Thus $\{a_n\}, \ \{b_n\}$ has a same limit $\mathrm{AGM}\,(a, b)$.

For example,

$$\mathrm{AGM}\left(1, \frac{1}{\sqrt{2}}\right) = 0.8472130847\cdots$$

and

n	a_n	b_n
0	1.0000000000	0.7071067811
1	0.8535533905	0.8408964152
2	0.8472249029	0.8472012667
3	0.8472130848	0.8472130847
4	0.8472130847	0.8472130847

Theorem(Gauss, 1809)

$$\pi = \frac{2\mathrm{AGM}(1, 1/\sqrt{2})^2}{1 - \sum_{n=0}^{\infty} 2^n c_n^2}$$

upper approximation

$$p_N = \frac{2a_N^2}{1 - \sum_{n=0}^{N} 2^n c_n^2}$$

p_0	4.	00000	00000	00000	00000
p_1	3.	18767	26427	12108	62720
p_2	3.	14168	02932	97653	29391
p_3	3.	14159	26538	95446	49600
p_4	3.	14159	26535	89793	23846
π	3.	14159	26535	89793	23846

Theorem (Gauss, 1808) Let $\mathrm{AGM}(1, \sqrt{1-k^2})$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be defined as above. Then,

$$\frac{1}{K(k)} = \frac{2}{\pi} \cdot \mathrm{AGM}(1, \sqrt{1-k^2}),$$

$$\frac{E(k)}{K(k)} = 1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2.$$

for $0 \leq k < 1$.

Moreover, the convergence is uniform on [0,1] by

$$AGM(1,0) = 0, \quad c_n = 2^{-n}$$

and

$$\lim_{k \uparrow 1} \frac{1}{K(k)} = 0, \quad \lim_{k \uparrow 1} \frac{E(k)}{K(k)} = 0.$$

To simplify the notation, we put $k := \sqrt{h}.$, and use the notations $\mathrm{AGM}(1, \sqrt{1-h})$, $\{a_n(h)\}$, $\{b_n(h)\}$, $\{c_n(h)\}$

Proposition 7.1.

$$\frac{1}{K(\sqrt{h})} = \frac{2}{\pi} \cdot \lim_{n\to\infty} a_n(h) = \frac{2}{\pi} \cdot \lim_{n\to\infty} b_n(h).$$

uniformly on $[0,1]$.

Proposition 7.2.

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} = 1 - \sum_{n=0}^{\infty} 2^{n-1} c_n(h)^2.$$

uniformly on $[0,1]$

8 Exact solutions for $dv_{xx} + v - v^2 - \tau = 0$

To get some feeling of deriving exact solutions of 1-d boundary value problems, let us shall first consider solutions of the following problem:

$$\begin{cases} d\ v_{xx} + v - v^2 - \tau = 0, \quad \text{in } (0,1), \\ v_x(0) = v_x(1) = 0, \\ v > 0, \quad \text{and} \quad v_x > 0, \quad \text{in } (0,1), \end{cases}$$

Let $v(0) = \alpha$, $v(1) = \beta$ $(\alpha < \beta)$. Multiplying the first equation by v_x and integrating it over $[0, x]$ with respect to x, we have

$$\frac{dv}{dx} = \sqrt{F(v)},$$

where

$$F(v) = \tfrac{1}{d}(\tfrac{2}{3}v^3 - v^2 - 2\tau v - \tfrac{2}{3}\alpha^3 + \alpha^2 + 2\tau\alpha).$$

Because of $v_x(1) = 0$, it must satisfy $F(\beta) = 0$. Let the remaining root of F be γ. Since $F(v) \geq 0$ for $\alpha \leq v \leq \beta$, γ must satisfy $\beta \leq \gamma$.

The equation (8) gives us

$$x = \sqrt{\frac{3d}{2}} \int_\alpha^v \frac{dw}{\sqrt{(w-\alpha)(\beta-w)(\gamma-w)}}. \tag{7}$$

Now, putting $w := \alpha + (\beta - \alpha)z^2$ in (7), we have

$$\sqrt{\frac{\gamma-\alpha}{6d}}x = \int_0^{\sqrt{\frac{v-\alpha}{\beta-\alpha}}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \tag{8}$$

where $k \in (0, 1)$ is defined by

$$k = \sqrt{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{9}$$

Letting $x = 1$ and $v = \beta$ in (11), we obtain

$$\sqrt{\frac{\gamma-\alpha}{6d}} = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = K(k). \tag{10}$$

Thus, we see from (8) and (10) that

$$K(k)x = \int_0^{\sqrt{\frac{v-\alpha}{\beta-\alpha}}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{11}$$

which implies

$$K(k)x = \mathrm{sn}^{-1}\left(\sqrt{\frac{v-\alpha}{\beta-\alpha}}, k\right).$$

Hence, we get

$$\mathrm{sn}(K(k)x, k) = \sqrt{\frac{v-\alpha}{\beta-\alpha}}.$$

Therefore, w have

$$v = \alpha + (\beta-\alpha)\mathrm{sn}^2\left(K(k)x, k\right). \tag{12}$$

Noting that

$$\alpha + \beta + \gamma = \frac{3}{2}, \tag{13}$$

and using the relations (9) and (10), we can represent α, β and γ by k,

$$\begin{cases} \alpha = \frac{1}{2} - 2dK(k)^2(k^2+1), \\ \beta = \frac{1}{2} + 2dK(k)^2(2k^2-1), \\ \gamma = \frac{1}{2} + 2dK(k)^2(2-k^2). \end{cases} \tag{14}$$

Thus, we get from (12)

$$\begin{aligned} v(x;k,d) &= \alpha + (\beta-\alpha)\mathrm{sn}^2\left(K(k)x, k\right) \\ &= \frac{1}{2} - 2dK(k)^2(k^2+1) + 6dkK(k)^2\mathrm{sn}^2\left(K(k)x, k\right). \end{aligned} \tag{15}$$

We also note that $\alpha\beta + \beta\gamma + \gamma\alpha = -3\tau$. Thus

$$\begin{aligned} \tau &= -\frac{1}{3}(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= -\frac{1}{4} + 4(k^4 - k^2 + 1)d^2K(k)^4. \end{aligned}$$

9 Stationary problem for Cahn-Hilliard eq.

9.1 Equation arising from Cahn-Hilliard equation

The following equation

$$\begin{cases} \varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\ \dfrac{du}{dx}(x) = 0, & x = 0, 1, \\ m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx, \end{cases} \tag{1}$$

is known as the stationary equation of the one-dimensional Cahn-Hilliard equation in the interval $(0,1)$, where ε is a positive parameter and f is the cubic polynomial

$$f(u) := u - u^3.$$

This equation is studied in an extensive literature including [20], [21], [22], [23], [24], [28], and [29] e.t.c.. Among other things, Grinfeld and Novick-Cohen [24] determine the number of solutions to (1) for each given (m, ε). They reduced the problem to a system of two equations for two unknown parameters and counted solutions to the system by applying some topological argument, called the transversality argument. From their result we can see how the solution bifurcates as (m, ε) varies.

Kosugi-Morita-Yotsutani [9] obtained an explicit form of each solution to (1) for every (m, ε) by using Jacobi elliptic functions $\mathrm{sn}(x,k)$ and $\mathrm{cn}(x,k)$, and the complete elliptic integrals $K(k)$, $E(k)$ and $\Pi(\nu, k)$.

We have recently recognized that the way of getting explicit form of each solution in [9] are very useful for related more difficult problems including a cell polarization problem.

Thus, we explain the method developed in in [9]. Some of proofs in it are a little bit complicated and hard to read since we did not find systematic way of controlling Jacob's complete elliptic integrals when we wrote the paper. We will show some improved proofs of systematic way.

Now we state the first result on the reduction of the equation (1) into the equations of two parameters (h, s), which involve the complete elliptic integrals.

9.2 Exact solutions and the global bifurcation

Proposition 9.1. *Given $\varepsilon > 0$ and $m \geq 0$, the non-local equation (1) has a monotone increasing solution $u(x)$ if and only if (h, s) satisfies the equation*

$$\varepsilon = \varepsilon(h, s), \quad m = \mathcal{M}(h, s), \quad (h, s) \in (0, 1) \times (0, 1) \subset \mathbb{R}^2, \tag{2}$$

where

$$\varepsilon(h, s) := \frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}, \tag{3}$$

$$\mathcal{M}(h, s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}. \tag{4}$$

For the solution (h, s) the equation (1) has a monotone increasing solution in the form

$$u(x) = \frac{\beta(h,s)(1-sh)\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha(h,s)\mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}, \tag{5}$$

where α and β are defined by

$$\alpha(h, s) := \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}},$$

$$\beta(h, s) := \frac{-hs^2 - 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}.$$

Analyzing (3) and (4), we can prove the following result which gives the number of the solutions to (2).

Theorem 9.1. *Put*

$$\eta_0(\xi) := \begin{cases} \sqrt{1-3\xi^2}/\pi, & \xi \in [0, 1/\sqrt{3}), \\ 0, & \xi \in [1/\sqrt{3}, 1]. \end{cases} \tag{6}$$

There exists a monotone decreasing continuous function

$\eta : [1/\sqrt{5}, 1] \ni \xi \to \eta(\xi) \in [0, \sqrt{2}/\sqrt{5}\pi]$

satisfying

$\eta(\xi) = \eta_0(\xi)$ *at* $\xi \in \{1/\sqrt{5}, 1\}$ *and* $\eta(\xi) > \eta_0(\xi)$ *for* $\xi \in (1/\sqrt{5}, 1)$

such that solutions (h, s) *of (2) exist if and only if*

$$\begin{aligned} (m, \varepsilon) \in \quad & \{(m, \varepsilon) : 0 < \varepsilon < \eta_0(m), \;\; 0 \le m \le 1/\sqrt{5}\} \\ & \cup \{(m, \varepsilon) : 0 < \varepsilon \le \eta(m), \;\; 1/\sqrt{5} < m < 1\}. \end{aligned}$$

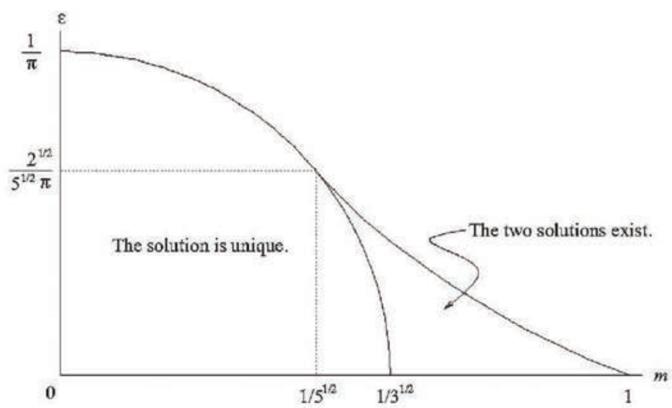
The number of the solutions depends on (m, ε) *as follows.*

If $\varepsilon \in (0, \eta_0(m))$ *for each* $m \in [0, 1/\sqrt{3}]$ *then the solution is unique.*

If $\varepsilon \in (\eta_0(m), \eta(m))$ *for each* $m \in (1/\sqrt{5}, 1)$ *then the equation (2) has two solutions.*

If ε … $m \in$

$(1/\sqrt{}$ …



We emphasize that the result for the number of the solutions of the above theorem was already obtained in [24]. However, by virtue of the explicit expression in Proposition 9.1, we can illustrate a bifurcation surface and bifurcation diagrams by simple numerics, which will be shown below. In addition we can investigate the limiting behavior of the solution as $\varepsilon \to 0$ by using the fact that α and β of Proposition 9.1 correspond to the minimum and maximum values of the solution of (1) for $x \in [0, 1]$ respectively. We obtain the following result:

Proposition 9.2. *Let* $(h, s) = (h(m, \varepsilon), s(m, \varepsilon))$ *be the unique solution to* (2) *for* $\varepsilon \in (0, \sqrt{1 - 3m^2}/\pi)$ *and* $m \in [0, 1/\sqrt{3})$. *Then*

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} h(m, \varepsilon) = 1, \qquad \lim_{\varepsilon \downarrow 0} s(m, \varepsilon) = 1, \\ &\lim_{\varepsilon \downarrow 0} \alpha(h(m, \varepsilon), s(m, \varepsilon)) = -1, \qquad \lim_{\varepsilon \downarrow 0} \beta(h(m, \varepsilon), s(m, \varepsilon)) = 1. \end{aligned} \tag{7}$$

Let $(h, s) = (h_i(m, \varepsilon), s_i(m, \varepsilon))$ $(i = 1, 2)$ *be solutions to* (2) *for* $\varepsilon \in (0, \eta(m))$ *and* $m \in [1/\sqrt{3}, 1)$ *satisfying* $h_1(m, \varepsilon) < h_2(m, \varepsilon)$. *Then*

$$\lim_{\varepsilon \downarrow 0} h_1(m, \varepsilon) = 1, \qquad \lim_{\varepsilon \downarrow 0} s_1(m, \varepsilon) = (3m^2 - 1)/(1 + m^2 + 2m\sqrt{2(1 - m^2)}),$$

$$\lim_{\varepsilon \downarrow 0} \alpha(h_1(m, \varepsilon), s_1(m, \varepsilon)) = \sqrt{2(1 - m^2)} - m, \tag{8}$$

$$\lim_{\varepsilon \downarrow 0} \beta(h_1(m, \varepsilon), s_1(m, \varepsilon)) = m, \tag{9}$$

$$\lim_{\varepsilon \downarrow 0} h_2(m, \varepsilon) = 1, \qquad \lim_{\varepsilon \downarrow 0} s_2(m, \varepsilon) = 1,$$

$$\lim_{\varepsilon \downarrow 0} \alpha(h_2(m, \varepsilon), s_2(m, \varepsilon)) = -1, \tag{10}$$

$$\lim_{\varepsilon \downarrow 0} \beta(h_2(m, \varepsilon), s_2(m, \varepsilon)) = 1. \tag{11}$$

Using the above proposition, we obtain the limit of the solution $u(x)$ as $\varepsilon \to 0$.

Theorem 9.2. *Put*

$$u_0(x;m) := \begin{cases} -1, & x \in [0,(1-m)/2), \\ 0, & x = (1-m)/2, \\ 1, & x \in ((1-m)/2, 1]. \end{cases}$$

(i) *Case* $m \in [0, 1/\sqrt{3})$*: let* $u = u(x; m, \varepsilon)$ *be the solution to* (1) *in the form of* (5) *for* $\varepsilon \in (0, \sqrt{1-3m^2}/\pi)$*. Then* $u(\cdot; m, \varepsilon)$ *uniformly converges to* $u_0(\cdot; m)$ *in any compact set of* $[0,1] \setminus \{(1-m)/2\}$*.*

(ii) *Case* $m \in [1/\sqrt{3}, 1)$*: let* $u = u_i(x; m, \varepsilon)$ $(i = 1, 2)$ *be the solutions to* (1) *which are given by substituting* $(h, s) = (h_i(m, \varepsilon), s_i(m, \varepsilon))$ $(i = 1, 2)$ *into* (5) *respectively for* $\varepsilon \in (0, \eta(m))$*. Then* $u_1(\cdot; m, \varepsilon)$ *uniformly converges to the constant* m *in any compact set of* $(0, 1]$ *while* $u_2(\cdot; m, \varepsilon)$ *uniformly converges to* $u_0(\cdot; m)$ *in any compact set of* $[0,1] \setminus \{(1-m)/2\}$*.*

In the next section we prove Proposition 9.1; that is, we derive (2) and show that non-trivial solutions to (2) are written in the form of (5) In the next of next section we prove several lemmas which are used to prove Theorem 9.1.

As a matter of fact, we show the existence of the implicit function $s = \ell(h; \varepsilon)$ that solves $\mathcal{E}(h, s) = \varepsilon$ and derive the necessary and sufficient condition that the equation $m = \mathcal{M}(h, \ell(h; \varepsilon))$ is solvable.

Although our proof is essentially based on the previous works [25, 26, 27], we need a lengthy computation. We divide the computation into the several steps.

9.3 Derivation of exact solutions

Assume that $u = u(x)$ is a monotone increasing solution to (1). Multiplying the differential equation in (1) by du/dx and calculating integration, we have

$$\left(\frac{du}{dx}\right)^2 = \frac{F(u)}{2\varepsilon^2}, \quad F(u) := u^4 - 2u^2 + 4au + p \tag{12}$$

for some constant $p \in \mathbb{R}$.

Let $\alpha := u(0)$ and $\beta := u(1)$. It then follows from

$$du/dx(0) = du/dx(1) = 0$$

that

$$F(\alpha) = F(\beta) = 0.$$

Thus

$$F(u) = (u - \alpha)(u - \beta)F_1(u),$$

where $F_1(u)$ is a quadratic polynomial.

Using $F(u(x)) > 0$ and the continuity of $u(x)$ for x, we have

$$F_1(u) < 0 \text{ for all } u \in (\alpha, \beta).$$

Hence there exist γ and δ such that

$$F_1(\gamma) = F_1(\delta) = 0 \text{ and } \gamma \leq \alpha < \beta \leq \delta.$$

Consequently,

$$F(u) = (u - \alpha)(u - \beta)(u - \gamma)(u - \delta), \quad \gamma \leq \alpha < \beta \leq \delta$$

and

$$\alpha + \beta + \gamma + \delta = 0, \quad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta = -2,$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\delta\gamma + \beta\delta\gamma = -4a, \quad \alpha\beta\gamma\delta = p.$$

Since

$$\frac{du}{dx} = \frac{\sqrt{F(u)}}{\sqrt{2}\varepsilon}$$

we have

$$x = \int_{\alpha}^{u(x)} \frac{\sqrt{2}\varepsilon}{\sqrt{(u-\alpha)(u-\beta)(u-\gamma)(u-\delta)}} du.$$

Here we notice $\gamma < \alpha < \beta < \delta$ by the fact that the integral must be finite.

Changing the variable

$$u = \frac{\gamma(\beta-\alpha)\tau^2 - \alpha(\beta-\gamma)}{(\beta-\alpha)\tau^2 - (\beta-\gamma)}, \quad \tau \in [0,1] \tag{13}$$

yields

$$x = \frac{2\sqrt{2}\varepsilon}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} \int_0^{\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}.} \frac{1}{\sqrt{1-k^2\tau^2}\sqrt{1-\tau^2}} d\tau,$$

where

$$k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\beta-\gamma)(\delta-\alpha)}}. \tag{14}$$

Taking $u(1) = \beta$ into account, we obtain

$$1 = \frac{2\sqrt{2}\varepsilon}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} K(k).$$

Thus, we get

$$K(k)x = \int_0^{\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}.} \frac{1}{\sqrt{1-k^2\tau^2}\sqrt{1-\tau^2}} d\tau,$$

which implies

$$K(k)x = \mathrm{sn}^{-1}\left(\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}, k\right)$$

Hence, we have

$$\mathrm{sn}(K(k)x, k) = \sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}.$$

Therefore, we obtain

$$u(x) = \frac{\gamma(\beta-\alpha)\,\mathrm{sn}^2(K(k)x,k) - \alpha(\beta-\gamma)}{(\beta-\alpha)sn^2(K(k)x,k) - (\beta-\gamma)}. \tag{15}$$

The integral (1) are also expressed by the Jacobi elliptic integrals with α, β, γ, and δ. Substituting (13) into it leads us to

$$\begin{aligned} m &= \int_0^1 u(x)\,dx = \int_\alpha^\beta \frac{\sqrt{2}\varepsilon u}{\sqrt{F(u)}}du \\ &= \sqrt{2}\varepsilon\left\{\int_\alpha^\beta \frac{u-\gamma}{\sqrt{F(u)}}du + \int_\alpha^\beta \frac{\gamma}{\sqrt{F(u)}}du\right\} \\ &= \sqrt{2}\varepsilon\left\{\frac{2(\alpha-\gamma)\Pi(\nu,k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} + \frac{2\gamma K(k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}}\right\}, \end{aligned}$$

where $\nu = -(\beta-\alpha)/(\beta-\gamma)$ and k is the same as in (14).

In consequence, if u is a monotone increasing solution to (1), then it is written in the form (15), where α, β, γ, δ, k, ν, ε, m, a, and p satisfy

$$\begin{aligned} &\gamma < \alpha < \beta < \delta, \\ &\alpha+\beta+\gamma+\delta = 0, \quad \alpha(\beta+\gamma+\delta)+\beta(\gamma+\delta)+\gamma\delta = -2, \\ &a = -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\delta\gamma+\beta\delta\gamma)/4, \quad p = \alpha\beta\gamma\delta, \\ &k = \sqrt{(\beta-\alpha)(\delta-\gamma)/(\beta-\gamma)(\delta-\alpha)}, \quad \nu = -(\beta-\alpha)/(\beta-\gamma), \\ &\varepsilon = \sqrt{(\delta-\alpha)(\beta-\gamma)}/2\sqrt{2}K(k), \\ &m = \sqrt{2}\varepsilon\left\{\frac{2(\alpha-\gamma)\Pi(\nu,k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} + \frac{2\gamma K(k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}}\right\}. \end{aligned}$$

Put $h = k^2$ and $s = -\nu/k^2$. Then the above relations allow the following explicit expressions with respect to (h,s):

$$\alpha = \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \qquad (16)$$

$$\beta = \frac{-hs^2 - 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \qquad (17)$$

$$\gamma = \frac{-hs^2 + 2(1+h)s - 3}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \qquad (18)$$

$$\delta = \frac{-hs^2 + 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \qquad (19)$$

$$\varepsilon = \frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}, \qquad (20)$$

$$m = \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}, \qquad (21)$$

$$a = \frac{2(hs^2 - 2sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}^3},$$

$$p = (3hs^2 - 2(1+h)s + 1)(-hs^2 - 2(1-h)s + 1)(hs^2 - 2(1+h)s + 3)$$
$$(hs^2 - 2(1-h)s - 1)(3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3)^{-2},$$

and $(h, s) \in (0,1) \times (0,1)$. Thus we obtain (2).

Conversely, if there exists a solution (h, s) to (2), then it follows from a straight calculation that (15) is a solution to (1).

10 Preliminary lemmas.

In this section we prove several lemmas which are used in the proof of Theorem 9.1.

First we compute the derivatives of $\mathcal{E}$ and $\mathcal{J} := \mathcal{M}/\mathcal{E}$. We frequently abbreviate the Jacobi elliptic functions $K(\sqrt{h})$, $E(\sqrt{h})$, and $\Pi(-sh, \sqrt{h})$ as K, E, and Π respectively.

Lemma 10.1. *Let*

$$\begin{aligned}\mathcal{J}(h,s) &:= \frac{\mathcal{M}(h,s)}{\mathcal{E}(h,s)}\\ &= -\frac{(hs^2-2(1+h)s+3)K(\sqrt{h})}{\sqrt{2s(1-s)(1-sh)}} + \frac{2\sqrt{2(1-s)(1-sh)}\Pi(-sh,\sqrt{h})}{\sqrt{s}}.\end{aligned}$$

The derivatives of $\mathcal{J}(h,s)$ and $\mathcal{E}(h,s)$ are

$$\frac{\partial}{\partial h}\mathcal{J}(h,s) = \frac{-(s^2h-2s+1)K}{2h\sqrt{2s(1-s)}\sqrt{1-sh}^3} + \frac{\sqrt{2}(s^2h-2(h+1)s-1)\,E}{4\,h\,(h-1)\sqrt{s(1-s)(1-sh)}}, \tag{22}$$

$$\frac{\partial}{\partial s}\mathcal{J}(h,s) = -\frac{\sqrt{2}\,(s^2h-2s+1)^2\,K}{4\sqrt{s(1-s)(1-sh)}^3} - \frac{\sqrt{2}\,E}{\sqrt{s(1-s)(1-hs)}}, \tag{23}$$

and

$$\begin{aligned}\frac{\partial}{\partial h}\mathcal{E}(h,s) = {}& \frac{\sqrt{s(1-s)}(h^2s^4+2(h^2-4h)s^3+4(1+2h)s^2-2(2+3h)s+3)}{h\sqrt{2(1-sh)}\sqrt{P(h,s)}^3 K}\\ &+ \frac{\sqrt{s(1-s)(1-sh)}E}{h(h-1)\sqrt{2P(h,s)}K^2},\end{aligned} \tag{24}$$

$$\frac{\partial}{\partial s}\mathcal{E}(h,s) = -\frac{3\,(s^2h-2sh+1)(s^2h-2s+1)(s^2h-1)}{\sqrt{2\,s\,(1-sh)\,(1-s)}\sqrt{P(h,s)}^3\,K}, \tag{25}$$

where

$$P(h,s) := 3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3. \tag{26}$$

Since the proof is a straightforward computation, we omit it.

The next lemma immediately follows from (25).

Lemma 10.2. *The derivative of $\mathcal{E}(h,s)$ with respect to s satisfies*

$$\frac{\partial}{\partial s}\mathcal{E}(h,s)\begin{cases} >0, & s\in(0,\sigma(h)),\quad h\in[0,1),\\ =0, & s=\sigma(h),\quad h\in[0,1),\\ <0, & s\in(\sigma(h),1),\quad h\in[0,1),\end{cases} \tag{27}$$

where $\sigma(h):=1/(1+\sqrt{1-h})$.

We next identify the domain in which $\mathcal{M}(h,s)$ is positive.

Lemma 10.3. *Let*

$$\mathfrak{M}^{+}:=\{(h,s):\mathcal{M}(h,s)>0\}\subset(0,1)\times(0,1).$$

Then $\mathfrak{M}^{+}=\{(h,s):0<s<\sigma(h),\ 0<h<1\}$. *Similarly,*

$$\mathfrak{M}^{0}:=\{(h,s):\mathcal{M}(h,s)=0\}=\{(h,s):s=\sigma(h),\ 0<h<1\},$$
$$\mathfrak{M}^{-}:=\{(h,s):\mathcal{M}(h,s)<0\}=\{(h,s):\sigma(h)<s<1,\ 0<h<1\}.$$

Proof. It follows from (22) and (23) that

$$\frac{d}{dh}\mathcal{J}(h,\sigma(h))=0,\qquad \frac{\partial}{\partial s}\mathcal{J}(h,s)<0.$$

Thus $\mathcal{J}(h,\sigma(h))=\mathcal{J}(0,\sigma(0))=0$ ($\forall h\in(0,1)$), which yields $\mathcal{J}(h,s)>0$ if $s\in(0,\sigma(h))$ and $\mathcal{J}(h,s)<0$ if $s\in(\sigma(h),1)$. With the aid of $\mathcal{M}(h,s)=\mathcal{E}(h,s)\mathcal{J}(h,s)$ and $\mathcal{E}(h,s)>0$, we obtain the lemma. □

Next we show that for each $\varepsilon>0$ the (non-empty) level set $\{(h,s)\in\mathfrak{M}^{+}\cup\mathfrak{M}^{0}:\mathcal{E}(h,s)=\varepsilon\}$ is given by a smooth curve.

Lemma 10.4. *There exist a unique solution* $h = h_\varepsilon$ *of*

$$\frac{1}{\sqrt{2(2-h)}\,K(\sqrt{h})} = \varepsilon, \quad h > 0, \tag{28}$$

and a unique curve $(h,s) = (h,\ell(h;\varepsilon))$ $(h \in [0,h_\varepsilon])$ *which gives the level set* $\{(h,s) \in \mathfrak{M}^+ \cup \mathfrak{M}^0 : \mathcal{E}(h,s) = \varepsilon\}$ *if and only if* $\varepsilon \in (0,1/\pi)$*, where* $\ell(\cdot;\varepsilon) \in C[0,h_\varepsilon] \cup C^\infty[0,h_\varepsilon)$ *and*

$$\begin{aligned}
&\ell(0;\varepsilon) = \left(1 - \sqrt{2(1-\varepsilon^2\pi^2)/(2+\varepsilon^2\pi^2)}\right)/2, \\
&\ell(h_\varepsilon;\varepsilon) = \sigma(h_\varepsilon).
\end{aligned} \tag{29}$$

Proof. By (27) in Lemma 10.2, we obtain

$$\mathcal{E}(h,s) < \mathcal{E}(h,\sigma(h)) \quad \forall (h,s) \in [0,1) \times [0,1] \setminus \{(h,s) : s = \sigma(h)\}.$$

On the other hand,

$$\mathcal{E}(h,\sigma(h)) = \frac{1}{\sqrt{2(2-h)}\,K(\sqrt{h})}, \quad \mathcal{E}(0,\sigma(0)) = \frac{1}{\pi}, \quad \mathcal{E}(h,\sigma(h)) \to 0 \ (h \uparrow 1)$$

and

$$\frac{d}{dh}\mathcal{E}(h,\sigma(h)) = \frac{2(1-h)K - (2-h)E}{2\sqrt{2}\,\sqrt{2-h}^{\,3}\,h\,(1-h)\,K^2} < 0.$$

Indeed this inequality follows from $(d/dh)(2(1-h)K - (2-h)E) = 3(E-K)/2 < 0$ with $2(1-0)K(0) - (2-0)E(0) = 0$. Thus there exists a unique solution $h = h_\varepsilon$ of (28) and $\mathcal{E}(h,\sigma(h)) > \varepsilon$ $(\forall h \in [0,h_\varepsilon))$ if and only if $\varepsilon \in (0,1/\pi)$. Since $\mathcal{E}(h,0) = 0$ and $\partial\mathcal{E}/\partial s(h,s) > 0$ $(\forall (h,s) \in \mathfrak{M}^+)$ (see (27) in Lemma 10.2), the equation $\mathcal{E}(h,s) = \varepsilon$ has a unique solution $s = \ell(h;\varepsilon) \in (0,\sigma(h)]$ for each $h \in [0,h_\varepsilon]$. Applying the standard argument for the smoothness of inverse functions yields the desired smoothness of $\ell(h;\varepsilon)$. On the other hand, we obtain (29) by the equality $\mathcal{E}(0,s) = 2\sqrt{2\,s\,(1-s)}/\pi\sqrt{4\,s^2 - 4\,s + 3}$. $\square$

Next we consider the composite function $\mathcal{M}(h,\ell(h;\varepsilon))$.

Lemma 10.5. *The critical point of* $\mathcal{M}(h, \ell(h;\varepsilon))$ *with respect to* h *is a solution to* $\varepsilon^2 = Q(h)$*, where*

$$Q(h) := \frac{(1-h)K(\sqrt{h})^2 + 2(h-2)E(\sqrt{h})K(\sqrt{h}) + 3E(\sqrt{h})^2}{2K(\sqrt{h})^3((1-h)(2-h)K(\sqrt{h}) - 2(h^2-h+1)E(\sqrt{h}))}. \quad (30)$$

Proof. Since $\mathcal{M}(h,s) = \mathcal{E}(h,s)\mathcal{J}(h,s)$ and $\mathcal{E}(h,\ell(h;\varepsilon)) = \varepsilon$, we have $\mathcal{M}(h,\ell(h;\varepsilon)) = \varepsilon\mathcal{J}(h,\ell(h;\varepsilon))$. From (22), (23), (24), and (25) in Lemma 10.1, it follows

$$\begin{aligned}\frac{d}{dh}\mathcal{J}(h,\ell(h;\varepsilon)) &= -\frac{\partial\mathcal{J}}{\partial s}\frac{\partial\mathcal{E}}{\partial h}\Big/\frac{\partial\mathcal{E}}{\partial s} + \frac{\partial\mathcal{J}}{\partial h} \\ &= \frac{\sqrt{2s(1-s)(1-sh)}^3 G(h,s)}{9h(h-1)(s^2h-1)(s^2h-2s+1)(s^2h-2sh+1)K^3},\end{aligned}$$

where $s = \ell(h;\varepsilon)$ and

$$\begin{aligned}G(h,s) = &-2\left\{(1-h)(2-h)K - 2(h^2-h+1)E\right\}K^3 \\ &+ \left\{(1-h)K^2 + 2(h-2)KE + 3E^2\right\}/\mathcal{E}(h,s)^2.\end{aligned}$$

Since $(1-h)(2-h)K - 2(h^2-h+1)E < 0$, the equation $G(h,\ell(h;\varepsilon)) = 0$ is equivalent to $\mathcal{E}(h,\ell(h;\varepsilon))^2 = Q(h)$. Therefore $(d/dh)\mathcal{J}(h,\ell(h;\varepsilon))$ vanishes if and only if $\varepsilon^2 = Q(h)$. Thus the lemma is proved. □

By applying the method explained in previous sections, iwe can show that Q is a monotone decreasing function with respect to h.

Lemma 10.6. $Q(h)$ *satisfies*

$$\frac{dQ}{dh}(h) < 0 \quad (h \in (0,1)), \qquad \lim_{h\downarrow 0} Q(h) = \frac{2}{5\pi^2}, \qquad \lim_{h\uparrow 1} Q(h) = 0.$$

Proof. Calculating the derivative of Q with respect to h, we have

$$\frac{dQ}{dh} = \frac{P_0(h, E/K)}{2h(1-h)((1-h)(2-h)K - 2(h^2-h+1)E)^2},$$

where

$$P_0(h, U) := 9(h^2-h+1)U^4 + 4(h-2)^3U^3 - 6(h-1)(h^2-7h+7)U^2 + 12(h-2)(h-1)^2U - 5(h-1)^3.$$

It thereby suffices to prove $P_0(h, E/K) < 0$. We can verify by applying the method explained in previous sections, $\square$

Remark. This lemma is Lemma 3.6 in Kosugi-Morita-Y. They had to use tricky and complicated argument to show $P_0(h, E/K) < 0$, since they did recognized the systematic way at that time.

Summarizing Lemmas 10.5 and 10.6, we obtain the following:

Proposition 10.1. *Let Q^{-1} be the inverse function of Q. The function $\mathcal{M}(h, \ell(h;\varepsilon))$ has a unique critical point $h = Q^{-1}(\varepsilon^2)$ in $(0, h_\varepsilon)$ if and only if $\varepsilon \in (0, \sqrt{2}/\sqrt{5}\pi)$. Moreover, if $\varepsilon \in (0, \sqrt{2}/\sqrt{5}\pi)$, then*

$$\frac{d}{dh}\mathcal{M}(h, \ell(h;\varepsilon)) > 0, \quad \forall h \in (0, Q^{-1}(\varepsilon^2)),$$
$$\frac{d}{dh}\mathcal{M}(h, \ell(h;\varepsilon)) < 0, \quad \forall h \in (Q^{-1}(\varepsilon^2), h_\varepsilon).$$

If $\varepsilon \in [\sqrt{2}/\sqrt{5}\pi, 1/\pi)$, then

$$\frac{d}{dh}\mathcal{M}(h, \ell(h;\varepsilon)) < 0, \quad \forall h \in (0, h_\varepsilon).$$

Proof. It is clear that

$$\frac{d}{dh}\mathcal{M}(h, \ell(h;\varepsilon)) = \varepsilon\frac{d}{dh}\mathcal{J}(h, \ell(h;\varepsilon)) = \frac{2\sqrt{2s(1-sh)(1-s)}^3(\varepsilon^2 - Q(h))G_1(h)}{9\varepsilon(hs^2-2s+1)(hs^2-2sh+1)(1-hs^2)(1-h)h}$$

holds, where $s = \ell(h;\varepsilon)$ and $G_1(h) = (1-h)(2-h)K - 2(h^2-h+1)E < 0$. Thus the assertion of the proposition immediately follows. $\square$

We can show the following lemmas, which are Lemmas 3.7, 3.8 and 3.9 in Kosugi-Morita-Y. The proof is a little bit long long, so we omit proofs of them..

To investigate the behavior of $\max\{\mathcal{M}(h, \ell(h; \varepsilon)) : 0 \leq h \leq h_\varepsilon\}$ for $\varepsilon \in (0, \sqrt{2}/\sqrt{5}\pi)$, we determine all the critical points.

Lemma 10.7. *Let Γ_c be the set of all the critical points of $\mathcal{M}(h, \ell(h; \varepsilon))$, that is,*

$$\Gamma_c := \bigcup_{\varepsilon \in (0, \sqrt{2}/\sqrt{5}\pi)} \{(h, s) \in \mathfrak{M}^+ : s = \ell(h; \varepsilon), h = Q^{-1}(\varepsilon^2)\}.$$

Then Γ_c is given by the nodal curve of the function

$$\begin{aligned} Q_1(h, s) := &(1-h)(hs^2 - 2s + 1)^2 K(\sqrt{h})^2 \\ &+ 2((h-2)s^2 + 2s - 1)(h^2 s^2 - 2hs - h + 2)K(\sqrt{h})E(\sqrt{h}) \\ &+ (3h^2 s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(1+h)s + 3)E(\sqrt{h})^2, \end{aligned} \tag{31}$$

that is, $\Gamma_c = \{(h, s) \in \mathfrak{M}^+ : Q_1(h, s) = 0\}$. Moreover, the equation $Q_1(h, s) = 0$ allows a unique solution $s = c(h)$ in the interval $(0, \sigma(h))$ for each $h \in [0, 1)$, hence

$$\Gamma_c = \{(h, s) : s = c(h), 0 < h < 1\}.$$

Next we investigate the function $c(h)$.

Lemma 10.8. *$c = c(h)$ is continuous and it satisfies*

$$\lim_{h \downarrow 0} c(h) = 1/2 - \sqrt{2}/4, \quad \lim_{h \uparrow 1} c(h) = 1, \quad \lim_{h \uparrow 1} \mathcal{M}(h, c(h)) = 1.$$

We next prove that $\mathcal{M}(h, c(h))$ is a monotone increasing function of h.

Lemma 10.9. *The derivative of $\mathcal{M}(h, c(h))$ with respect to h is positive for $h \in (0, 1)$.*

11 Related nonlocal problems

We are interested in the global structure of all solutions of several nonlocal nonlinear boundary problems arising in various fileds. We show examples.

11.1 Oseen's spiral flow

The first problem is related with the Oseen's spiral flow [17]. Find a function $U(x)$ such that

$$
\text{(O)} \quad \begin{cases} U_{xx} + AU = U^2 - \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} U(x)^2 \, \mathrm{d}x, \quad x \in (-\pi, \pi), \\ U(-\pi) = U(\pi), \quad U_x(-\pi) = U_x(\pi), \\ \displaystyle\int_{-\pi}^{\pi} U(x) \, \mathrm{d}x = 0, \end{cases}
$$

for arbitrarily fixed A.

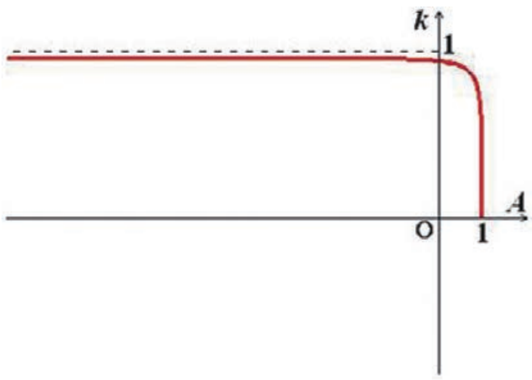
It is easily seen that $U \equiv 0$ is the trivial solution of the above problem for any fixed A. Okamoto [16] started to investigate the global bifurcation structure of this problem. Moreover, Ikeda-Mimura-Okamoto [7] obtained the asymptotic shape of solutions as $A \to -\infty$.

Ikeda-Kondo-Okamoto-Yotsutani [6] have parameterized all solutions (A, U) of (O) in terms of the elliptic functions, and clarified the global bifurcation structure by the following Theorems 11.1 and 11.2.

Theorem 11.1. *All the solution* (A, U) *of* (O) *are parameterized by* $\left\{\left(n^2A(k), n^2U(nx-x_0; A(k))\right) : 0 < k < 1, -\pi < x_0 \leq \pi, n = 1, 2, 3\cdots\right\}$,
where

$$\begin{aligned} A(k) &:= \frac{4K(k)}{\pi^2}\left(3E(k) + (k^2-2)K(k)\right), \\ U(x; A(k)) &:= -\frac{6k^2K(k)^2}{\pi^2}\,\mathrm{cn}^2\left(\frac{K(k)}{\pi}x, k\right) \\ &\quad + \frac{6K(k)}{\pi^2}\{E(k) - (1-k^2)K(k)\}. \end{aligned}$$

Theorem 11.2. *The function* $A(k)$ *is strictly monotone decreasing in* $k \in (0, 1)$*. It also satisfies* $\lim_{k\to 0} A(k) = 1$ *and* $\lim_{k\to 1} A(k) = -\infty$*.*



11.2 One-dimensional Ginzburg-Landau models

The second problem is related with structure of stationary solutions in S^1 of the Ginzburg-Landau equation.

Find a function $u(x)$ such that

$$
\text{(P)} \quad \begin{cases} u_{xx} - \dfrac{C^2}{u^3} + \lambda(1-u^2)u = 0 \quad \text{in } [-\pi, \pi], \\ C := 2m\pi \left\{ \displaystyle\int_{-\pi}^{\pi} \frac{1}{u^2} dx \right\}^{-1}, \\ u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi), \\ u > 0 \ \text{ in } [-\pi, \pi], \end{cases}
$$

where m is a given integer and λ is a bifurcation parameter.

The structure of solutions is similar to that of Oseen's spiral flow, though the analysis is more difficult. Kosugi-Morita-Yotsutani [8] have clarified the global bifurcation structure of this problem. We omit the detail here.

11.3 Closed plane elastic curves with area constraint

The third problem is related to closed plane elastic curves with area constraint for given the length L and area M, which K.Watanabe [19] started to investigate.

For given $L > 0$ and $M > 0$ with $L^2 - 4\pi M > 0$, find a function $\kappa(s)$ such that

$$
\text{(E)} \begin{cases} \{\kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa\}_s = 0, \quad \text{in } [0, L], \\ \mu := \dfrac{1}{L^2 - 4\pi M} \left\{ M \displaystyle\int_0^L \kappa(s)^3 ds - \frac{L}{2} \int_0^L \kappa(s)^2 ds \right\}, \\ \kappa(0) = \kappa(L), \quad \kappa_s(0) = \kappa_s(L), \\ \displaystyle\int_0^L \kappa(s) ds = 2\pi. \end{cases}
$$

Murai-Matsumoto-Yotsutani [13] have completely clarified the global bifurcation structure of this problem, though we need terribly complicated calculations and arguments.

The following reference is a just memo.

Thee are lot of errors, overlaps, and Japanese. Sorry!

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