Two topics in the theory of reaction-diffusion equations

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Time periodic solutions of a degenerate parabolic problem

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Plan of the lecture

- Statement of the problem and motivations
- Definitions and preliminary results
- Main results:
  - Existence of periodic solutions
  - The role of boundary conditions: Dirichlet versus Neumann
- Monotonicity methods
- Sketch of the proofs:
  - Construction of periodic sub- and supersolutions
  - The autonomous case as a hint
- Multiplicity and support properties (overview)

Problem: Existence, multiplicity and support properties of time periodic solutions of the parabolic equation

$$\partial_t u = \Delta u^m + au \quad \text{in} \;\; \mathbb{R} \times \Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $m > 1$.

Main features of the problem:

(a) *since $m > 1$, the equation is degenerate* (it reduces to the **porous medium equation** if $a \equiv 0$);

(b) *$a : \mathbb{R} \times \Omega \to \mathbb{R}$ is time periodic: there exists $T > 0$ such that*

$$a(t+T, x) = a(t, x) \text{ for any } (t, x) \in \mathbb{R} \times \Omega\,;$$

(c) *$a(t, \cdot)$ changes sign in Ω for any $t \in \mathbb{R}$.*

Motivations

Suppose that a population lives in a habitat Ω, in which there exists a favourable region Ω^+ where the population flourishes, whereas in the complementary region $\Omega^- := \Omega \setminus \Omega^+$ the population perishes. Then, depending on the relative size of Ω^+ and Ω^-, the dispersal and growth rates of the population and the conditions at the boundary $\partial\Omega$, the population either may die out, or tend to a stable distribution, or grow explosively. A classical problem in population dynamics is to determine which of these three possibilities will actually occur (see [O]).

In the model proposed in [GM, GN, N] the population density $u = u(t, x)$ satisfies equation (1).

Choosing $m > 1$ models a tendency to avoid crowding (see [O, SKT]). Clearly, the local growth rate $a = a(t, x)$ must be positive in Ω^+, and negative in Ω^-; therefore, the problem is *of indefinite type*. If time periodic fluctuations of the habitat (*e.g.*, seasonal fluctuations, or tidal effects) are taken into account, the local growth rate must be periodic.

Assumptions

The function

$$a : \mathbb{R} \times \Omega \to \mathbb{R}, \quad (t, x) \mapsto a(t, x)$$

is continuous and satisfies the following assumptions:

$$(A) \quad \begin{cases} a(\cdot, x) \text{ is } T\text{-periodic: there exists } T > 0 \text{ such that} \\ a(t+T, x) = a(t, x) \text{ for any } (t, x) \in \mathbb{R} \times \Omega\,; \\ a(\cdot, x) \text{ is Lipschitz continuous in } \mathbb{R}, \text{ uniformly for } \in \overline{\Omega}\,; \\ a(t, \cdot) \text{ changes sign in } \Omega, \text{ for any } t \in \mathbb{R}\,; \\ a(t, \cdot) \text{ is Hölder continuous in } \Omega, \text{ uniformly for } t \in \mathbb{R}\,. \end{cases}$$

The case where a does not depend on time,

$$a(t, x) \equiv a(x) \qquad (t \in \mathbb{R}, x \in \Omega) \tag{2}$$

is called the *autonomous case*.

Equation (1) will be complemented either with Dirichlet, or with Neumann homogeneous boundary conditions. Therefore, we are interested in time T-periodic solutions of the following systems:

$$(D) \qquad \begin{cases} \partial_t u = \Delta u^m + au & \text{in } \mathbb{R} \times \Omega \\ u = 0 & \text{in } \mathbb{R} \times \partial\Omega\,, \end{cases}$$

$$(N) \qquad \begin{cases} \partial_t u = \Delta u^m + au & \text{in } \mathbb{R} \times \Omega \\ \dfrac{\partial u^m}{\partial n} = 0 & \text{in } \mathbb{R} \times \partial\Omega \end{cases}$$

(where n denotes the outer unit normal at $\partial\Omega$, which is smooth by assumption).

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Time periodic solutions for a class of degenerate parabolic problems, Houston J. Math. **21** (1995), 367-394.

For the autonomous case:

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Definition of solution, sub- and supersolution

Since $m > 1$, equation (1) reduces to the porous medium equation if $a \equiv 0$; accordingly, solutions are defined in the following weak sense.

Definition 1

A function u is called a *solution* of problem (D) (respectively, of problem (N)) in $[0, \tau]$ $(\tau > 0)$, if :

(i) $u \in C([0, \tau); L^1(\Omega)) \cap L^\infty(Q_\tau)$, where $Q_\tau := (0, \tau] \times \Omega$;

(ii) for every $\eta \in C^2(\overline{Q}_\tau)$, $\eta \geq 0$ such that $\eta = 0$ (respectively, $\frac{\partial \eta}{\partial n} = 0$) in $[0, \tau] \times \partial\Omega$ there holds

$$\int_\Omega u(\tau, x)\eta(\tau, x)\, dx - \int\int_{Q_\tau} \{u\partial_t \eta + u^m \Delta\eta\}\, dtdx = \tag{3}$$

$$= \int_\Omega u(0, x)\eta(0, x)\, dx + \int\int_{Q_\tau} au\eta\, dtdx\,.$$

A *subsolution* $\underline{u}$ (respectively, a *supersolution* $\overline{u}$) is defined by replacing in (3) " = " by " ≤ " (respectively, by " ≥ ").

Definition 2

A function u is called a *T- periodic solution* of problem (D) (respectively, of problem (N)), if it is a solution in $[0, T]$ such that

$$u(0,\cdot) = u(T,\cdot) \quad a.e. \text{ in } \Omega .$$

A function $\underline{u}$ is called a *T- periodic subsolution* of problem (D) (respectively, of problem (N)), if it is a subsolution in $[0, T]$ such that

$$\underline{u}(0,\cdot) \le \underline{u}(T,\cdot) \quad a.e. \text{ in } \Omega .$$

A function $\overline{u}$ is called a *T- periodic supersolution* of problem (D) (respectively, of problem (N)), if it is a supersolution in $[0, T]$ such that

$$\overline{u}(0,\cdot) \ge \overline{u}(T,\cdot) \quad a.e. \text{ in } \Omega .$$

In the autonomous there exist *stationary solutions* of problems (D) and (N), which can be regarded as a special case of T-periodic solutions. They solve either problem:

$$\begin{cases} \Delta u^m + au = 0 & \text{in } \mathbb{R} \times \Omega \\ \text{either } u = 0, \text{ or } \dfrac{\partial u}{\partial n} = 0 & \text{in } \mathbb{R} \times \partial\Omega\,. \end{cases}$$

Definition 3

A function $u \in L^\infty(\Omega)$ is called a *stationary solution* of problem (D) (respectively, of problem (N)), if for every $\eta \in C^2(\overline{\Omega})$, $\eta \geq 0$ such that $\eta = 0$ (respectively, $\frac{\partial \eta}{\partial n} = 0$) on $\partial\Omega$ there holds

$$-\int_\Omega u^m \Delta\eta \, dx = \int_\Omega au\eta \, dx\,. \tag{4}$$

Stationary subsolutions and supersolutions are defined by replacing in (4) " = " by " ≤ " (respectively, by " ≥ ").

To study problems (D) and (N), we need results concerning the associated *initial-boundary value problems:*

$$(CD)\qquad \begin{cases} \partial_t u = \Delta u^m + au & \text{in } (0,\infty)\times\Omega \\ u = 0 & \text{in } (0,\infty)\times\partial\Omega \\ u = u_0 & \text{in } \{0\}\times\Omega\,, \end{cases}$$

$$(CN)\qquad \begin{cases} \partial_t u = \Delta u^m + au & \text{in } (0,\infty)\times\Omega \\ \dfrac{\partial u^m}{\partial n} = 0 & \text{in } (0,\infty)\times\partial\Omega \\ u = u_0 & \text{in } \{0\}\times\Omega\,. \end{cases}$$

We always assume that $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *a.e.* in Ω.

Let us state the following definition.

Definition 4

A function u is called a *solution* of the initial-boundary value problem (CD) (respectively, of problem (CN)) in $[0, \tau]$ $(\tau > 0)$, if it is a solution of problem (D) (respectively, of problem (N)) in $[0, \tau]$ such that

$$u(0, \cdot) = u_0 \quad a.e. \text{ in } \Omega\,.$$

Sub- and *supersolutions of problems* (CD) *and* (CN) are defined in an obvious way.

Preliminary results

In the following we shall need several results concerning solutions of the initial-boundary value problems (CD) and (CN); only those relative to (CD) are explicitly stated.

Theorem 5

For any $u_0 \in L^\infty(\Omega)$ there exists $t_0 > 0$ (only depending on the norm $\|u_0\|_{L^\infty(\Omega)}$) such that in $[0, t_0)$ there exists a solution of problem (CD).

Theorem 6

Let $\underline{u}$ be a subsolution and $\overline{u}$ be a supersolution of problem (CD) in $[0, \tau]$ $(\tau > 0)$, such that $\underline{u}(0, \cdot)$ and $\overline{u}(0, \cdot)$ belong to $L^\infty(\Omega)$. Then

$$\underline{u}(0, \cdot) \leq \overline{u}(0, \cdot) \ \text{ a.e. in } \Omega \quad \Longleftrightarrow \quad \underline{u} \leq \overline{u} \ \text{ a.e. in } Q_\tau .$$

Theorem 7

For any $u_0 \in L^\infty(\Omega)$ the solution of problem (CD) is global.

Proof. For any $\tau > 0$ set $\overline{a} := sup_{(t,x)\in Q_\tau} a(t,x)$. Then choose in Theorem 6

$$\underline{u} = u\,, \quad \overline{u} = \|u_0\|_{L^\infty(\Omega)}\, e^{\overline{a}t}\,.$$

□

Theorem 8

Let u be the global solution of problem (CD) given by Theorem 7, and let $0 < \tau < \overline{\tau}$. Then there exists a map ω_τ, continuous and nondecreasing in $[0,\infty)$ (only depending on τ and $\|u\|_{L^\infty(Q_{\overline{\tau}})}$), such that $\omega_\tau(0) = 0$ and

$$|u(t_2,x_2) - u(t_1,x_1)| \le \omega_\tau\left(|x_2 - x_1| + \sqrt{|t_2 - t_1|}\right)$$

for any $(t_1,x_1),(t_2,x_2) \in [\tau,\overline{\tau}] \times \overline{\Omega}$.

Existence results

Define

$$\Omega^+ := \left\{ x \in \Omega : \int_0^T a(t,x)\, dx > 0 \right\}, \tag{5}$$

$$\Omega^+ = \bigcup_{k \in \mathcal{M}} \Omega_k^+ \qquad (\mathcal{M} \subseteq \mathbb{N}).$$

Concerning existence of T-periodic solutions of problem (D), the following result can be proven.

Theorem 9

Let the set Ω^+ be nonempty. Then there exists a nontrivial, nonnegative T-periodic solution of problem (D).

To prove a similar result for problem (N), a new condition appears, which is not needed for problem (D).

Theorem 10

Let the set Ω^+ be nonempty, and

$$\int\int_{Q_T} a(t,x)\,dtdx < 0\,. \tag{6}$$

Then there exists a nontrivial, nonnegative T-periodic solution of problem (N).

Condition (6) is "almost necessary" for problem (N):

Theorem 11

Let there exist a T-periodic solution of problem (N), positive in $\overline{Q}_T$. Then

$$\int\int_{Q_T} a(t,x)\,dtdx \le 0\,, \tag{7}$$

and equality holds if and only if a does not depend on x.

Monotonicity methods

Let E be an ordered Banach space, $U \subseteq E$ any subset. A map $S : U \to U$ is called *order preserving* if

$$u, v \in U, \;\; u \leq v \quad \Rightarrow \quad S(u) \leq S(v)\,.$$

In addition, $\underline{u}$, $\overline{u} \in U$ are called a *subsolution*, respectively a *supersolution* of the *fixed point equation*

$$S(u) = u\,, \tag{8}$$

if

$$\underline{u} \leq S(\underline{u})\,, \quad \overline{u} \geq S(\overline{u})\,.$$

Proposition 1

Let $S : U \subseteq E \to U$ *be continuous and order preserving. Let* $\underline{u} \leq \overline{u}$ *be a subsolution, respectively a supersolution of equation* (8) *such that*

$$[\underline{u}, \overline{u}] := \{u \in E : \underline{u} \leq u \leq \overline{u}\} \subseteq U\,.$$

Let the set $S([\underline{u}, \overline{u}])$ *be relatively compact in* E*. Define*

$$v_0 := \underline{u}\,, \quad v_{k+1} := S(v_k)\,,$$

$$w_0 := \overline{u}\,, \quad w_{k+1} := S(w_k) \qquad (k \in \mathbb{N})\,.$$

Then

$$v_{k+1} \to u_*\,, \quad w_{k+1} \to u^*\,,$$

where u_*, u^* *are solutions of equation* (8) *such that* $u_* \leq u^*$.

Let us apply the above abstract result to problems (D) and (N). To this purpose, consider the *semiflow* S_t associated with the solution $u = u(t; u_0)$ of the initial-boundary value problems (CD), respectively (CN), namely

$$S_t : L^\infty(\Omega) \to L^\infty(\Omega)\,, \quad S_t(u_0) := u(t; u_0) \qquad (t > 0)\,.$$

Observe that for $t = T$ the operator S_T is the *Poincaré map*. By Theorems 5-8, the map S_t is

1. defined for any $t > 0$ (Theorems 5 and 7),
2. order preserving (Theorem 6),
3. compact (Theorem 8). In fact, for any fixed $M > 0$ the set $\{S_t u_0 \mid \|u_0\|_\infty \leq M\}$ is bounded in $L^\infty(\Omega)$ and consists of equicontinuous functions, thus the claim follows by the Ascoli-Arzelá Theorem.

In addition, the proof of Theorem 5 shows that S_t is continuous.

Now we can prove the following

Theorem 12

Let $\underline{u}$ be a T-periodic subsolution and $\overline{u}$ a T-periodic supersolution of problem (D) (respectively (N)), such that

$$\underline{u}(0,\cdot) \leq \overline{u}(0,\cdot) \text{ a.e. in } \Omega . \tag{9}$$

Then there exist T-periodic solutions u_*, u^* of problem (D) (respectively (N)), such that

$$\underline{u} \leq u_* \leq u^* \leq \overline{u} \text{ a.e. in } Q_T .$$

A fixed point $\tilde{v}$ of the Poincaré map S_T - namely, a function $\tilde{v} \in L^\infty(\Omega)$ such that $S_T(\tilde{v}) = \tilde{v}$ - is a T-periodic solution. Hence we must prove that there exists two fixed points $u_* \leq u^*$ of S_T; this is made by using Proposition 1.

Proof. By Theorem 6 there holds

$$S_T(\underline{u}(0,\cdot)) \geq \underline{u}(T,\cdot)$$

(the above inequality and the following ones hold *a.e.* in Ω), Then by Definition 2 we have that

$$S_T(\underline{u}(0,\cdot)) \geq \underline{u}(0,\cdot)\,,$$

whence

$$S_{(k+1)T}(\underline{u}(0,\cdot)) \geq S_{kT}(\underline{u}(0,\cdot)) \quad \text{for any } k \in \mathbb{N}\,.$$

Similarly,

$$S_T(\overline{u}(0,\cdot)) \leq \overline{u}(T,\cdot) \leq \overline{u}(0,\cdot)\,,$$

whence

$$S_{(k+1)T}(\overline{u}(0,\cdot)) \leq S_{kT}(\overline{u}(0,\cdot)) \quad \text{for any } k \in \mathbb{N}\,.$$

Moreover, since S_{kT} is order preserving, from inequality (9) we get

$$S_{kT}(\underline{u}(0,\cdot)) \leq S_{kT}(\overline{u}(0,\cdot)) \quad \text{for any } k \in \mathbb{N}.$$

It has been already observed that S_T satisfies the assumptions made in Proposition 1 on the map S. Then by the same proposition there exist $v_*,\, v^* \in L^\infty(\Omega)$ such that

(i) $\quad S_{kT}(\underline{u}(0,\cdot)) \to v_*, \;\; S_{kT}(\overline{u}(0,\cdot)) \to v^*$ as $k \to \infty$ in $L^\infty(\Omega)$;

(ii) $\quad S_T(v_*) = v_*,\, S_T(v^*) = v^*$ *a.e.* in Ω;

(iii) $\quad v_* \leq v^*$ *a.e.* in Ω.

Defining

$$u_*(t,\cdot) := S_t(v_*)\,, \quad u^*(t,\cdot) := S_t(v^*) \;\; \text{for any } t > 0$$

the result follows. □

Remark: In the autonomous case Proposition 1 can be used similarly to prove existence of stationary solutions $u_*,\, u^*$ of problems (D) and (N) such that $u_* \leq u^*$ *a.e.* in Ω. The whole point is that $S_{t+s} = S_t \circ S_s$ for any $s, t > 0$. Hence, if $\underline{u}$ is a stationary subsolution, the map from $\mathbb{R}$ to $L^\infty(\Omega)$, $t \mapsto S_t(\underline{u})$ is nondecreasing. Similarly, if $\overline{u}$ is a stationary supersolution, the map $t \mapsto S_t(\overline{u})$ is nonincreasing.

How to proceed?

Now we want to apply Theorem 12 to prove the existence results. To this purpose, we must construct T-periodic sub- and super solutions of problems (D) and (N).

As already remarked, stationary solutions, which exists in the autonomous case, can be regarded as particular T-periodic solutions. Hence it is useful to discuss first this simpler case, see:

- *C. Bandle, M.A. Pozio & A. Tesei, The asymptotic behavior of the solutions of degenerate parabolic equations*, Trans. Amer. Math. Soc. **303** (1987), 487-501.

- *C. Bandle, M.A. Pozio & A. Tesei, Existence and uniqueness of solutions of nonlinear Neumann problems*, Math. Z. **199** (1988), 257-278.

Let $a(t,x) = a(x)$. By assumption (A),

$$\Omega^+ := \{x \in \Omega \mid a(x) > 0\} \neq \emptyset, \quad \Omega^- := \{x \in \Omega \mid a(x) < 0\} \neq \emptyset.$$

The autonomous case: Dirichlet boundary conditions

Remark: If u is a nonnegative stationary solution of problem (D), then $v := u^m$ is a classical solution of the problem

$$(DS) \qquad \begin{cases} \Delta v + a v^{\frac{1}{m}} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \,. \end{cases}$$

Theorem 13

Let $\Omega^+ \neq \emptyset$. Then there exists a stationary solution of problem (D) which is positive in Ω^+.

Remark: It can be proven that the solution mentioned in Theorem 13 is unique. However, in general there exist other nontrivial stationary solutions of problem (D); their multiplicity depends on the number of maximal connected components of Ω^+.

Constructing subsolutions

The proof of Theorem 13 relies on Theorem 12, and consists in constructing *small subsolutions* and *large supersolutions* .

Small subsolutions: let B_k be any open ball such that $\overline{B}_k \subset \Omega_k^+$ $(k \in M)$. Define

$$\underline{a}_k := \min_{x \in \overline{B}_k} a(x) > 0 \,,$$

and consider the eigenvalue problem

$$\begin{cases} \Delta \xi_k + \lambda_k \underline{a}_k \xi_k = 0 & \text{in } B_k \\ \xi_k = 0 & \text{on } \partial B_k \\ \xi_k > 0 & \text{in } B_k \,, \end{cases}$$

with $\|\xi_k\|_\infty = 1$. It is easily seen that the function

$$\underline{v}_{k,\rho} := \begin{cases} (\rho \xi_k)^{\frac{1}{m}} & \text{in } B_k \\ 0 & \text{in } \Omega \setminus \overline{B}_k \end{cases}$$

is a stationary subsolution of (D) for any $\rho > 0$ sufficiently small.

In fact, for every $\eta \in C^2(\overline{\Omega})$, $\eta \geq 0$ in Ω, $\eta = 0$ on $\partial\Omega$ there holds (by setting $\underline{v} \equiv \underline{v}_{k,\rho}$ for simplicity)

$$\begin{aligned}
\int_\Omega \{\underline{v}^m \Delta\eta + a\underline{v}\eta\}\, dx &= \int_{B_k} \{\rho\xi_k \Delta\eta + a(\rho\xi_k)^{\frac{1}{m}}\eta\}\, dx = \\
&= \int_{B_k} \{\rho\Delta\xi_k + a(\rho\xi_k)^{\frac{1}{m}}\}\,\eta\, dx - \rho\int_{\partial B_k} \frac{\partial\xi_k}{\partial n}\eta\, dx \geq \\
&\geq \int_{B_k} \{-\lambda_k \underline{a}_k \rho\xi_k + a(\rho\xi_k)^{\frac{1}{m}}\}\,\eta\, dx \geq \\
&\geq \underline{a}_k \int_{B_k} \{1 - \lambda_k (\rho\xi_k)^{1-\frac{1}{m}}\}(\rho\xi_k)^{\frac{1}{m}}\,\eta\, dx > 0,
\end{aligned}$$

$$\text{if} \quad \rho < \lambda_k^{\frac{1}{m}-1}.$$

Constructing supersolutions (Dirichlet case)

Large supersolutions: Let B be any open ball such that $B \subset \overline{\Omega}$.
Consider the eigenvalue problem

$$\begin{cases} \Delta\xi + \lambda\xi = 0 & \text{in } B \\ \xi = 0 & \text{on } \partial B \\ \xi > 0 & \text{in } B\,. \end{cases}$$

It can be proven that for any $M > 0$ there exists $\sigma_0 > 0$ such that for any $\sigma > \sigma_0$ the function

$$\overline{v}_\sigma := (\sigma\xi)^{\frac{1}{m}} \tag{10}$$

is a stationary supersolution of problem (D), and $\overline{v}_\sigma \geq M$ in Ω.

Therefore, there exist *arbitrarily large* stationary supersolutions of problem (D).

In fact, set

$$\varepsilon := \min_{x\in\overline{\Omega}} \xi(x) > 0\,, \quad \overline{a} := \max_{x\in\overline{\Omega}} a(x) < \infty\,.$$

For every $\eta \in C^2(\overline{\Omega})$, $\eta \geq 0$ in Ω, $\eta = 0$ on $\partial\Omega$ there holds (by setting $\overline{v} \equiv \overline{v}_\sigma$ for simplicity)

$$\begin{aligned}\int_\Omega \{\overline{v}^m \Delta\eta + a\overline{v}\eta\}\,dx &= \int_\Omega \{\sigma\xi\Delta\eta + a\,(\sigma\xi)^{\frac{1}{m}}\,\eta\}\,dx = \\ &= \int_\Omega \{\sigma\Delta\xi + a\,(\sigma\xi)^{\frac{1}{m}}\}\,\eta\,dx = \\ &= \int_\Omega \{-\lambda\sigma\xi + a\,(\sigma\xi)^{\frac{1}{m}}\}\,\eta\,dx \leq \\ &\leq \int_\Omega \{\overline{a} - \lambda\,(\sigma\xi)^{1-\frac{1}{m}}\}\,(\sigma\xi)^{\frac{1}{m}}\,\eta\,dx < 0\end{aligned}$$

if $\quad \sigma > \frac{1}{\varepsilon}\,(\frac{\overline{a}}{\lambda})^{\frac{m}{m-1}}$, which implies $\overline{a} - \lambda\,(\sigma\varepsilon)^{1-\frac{1}{m}} < 0$.

Moreover, there holds $\overline{v}_\sigma \geq M$ if $\sigma \geq \frac{M^m}{\varepsilon}$.

Proof of Theorem 13

Proof.

The function

$$\underline{v}_\rho := \begin{cases} (\rho\xi_k)^{\frac{1}{m}} & \text{in } B_k \text{ for any } k \in \mathcal{M} \\ 0 & \text{in } \Omega \setminus \cup_{k\in\mathcal{M}} \overline{B}_k \end{cases} \tag{11}$$

is a stationary subsolution of (D) for any $\rho > 0$ sufficiently small, and $\underline{v}_\rho \leq \overline{v}_\sigma$ for any $\rho > 0$ sufficiently small and $\sigma > 0$ sufficiently large. The resulting stationary solution of problem (D), which exists by Proposition 1, is nonnegative and different from zero in each Ω_k^+ by construction, thus it is positive in Ω^+ by the maximum principle. □

Remark: (i) We can construct subsolutions which vanish in some Ω_k^+; (ii) by the maximum principle, either $u > 0$, or $u \equiv 0$ in each Ω_k^+. Therefore, we can wonder whether nonnegative stationary solutions exist, which vanish in some Ω_k^+. Such solutions actually exist, thus the multiplicity of solutions depends on the structure of the set Ω^+.

The autonomous case: Neumann boundary conditions

Remark: If u is a nonnegative stationary solution of problem (N), then $v := u^m$ is a classical solution of the problem

$$(NS) \qquad \begin{cases} \Delta v + a v^{\frac{1}{m}} = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

The counterpart of Theorem 13 is the following

Theorem 14

Let the set Ω^+ be nonempty. A stationary solution of problem (N) positive in Ω^+ exists if and only if

$$\int_\Omega a(x)\, dx < 0\,. \tag{12}$$

Constructing supersolutions (Neumann case)

Large supersolutions: For any $\varepsilon \in (0,1)$, $k > 0$ define

$$S_{\varepsilon,k} := \{ w \in C^{\alpha}(\overline{\Omega}) \mid \int_{\Omega} w(x)\, dx = 0 \,, \ \|w\|_{\infty} < \varepsilon k \} \ .$$

Consider the problem

$$\begin{cases} -\Delta\phi = G(\cdot, k+w) & \text{in } \Omega \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{13}$$

where $k > 0$, $w \in S_{\varepsilon,k}$ and

$$G(x, k+w(x)) := a(x)[k+w(x)]^{\frac{1}{m}} - \frac{1}{|\Omega|} \int_{\Omega} a(y)[k+w(y)]^{\frac{1}{m}}\, dy \qquad (x \in \overline{\Omega}) \, .$$

The following lemma can be proven.

Lemma 2

For any $\varepsilon \in (0,1)$ *there exists* $\overline{k} = \overline{k}(\varepsilon) > 0$ *such that for every* $k > \overline{k}$ *problem* (13) *a unique solution* $\phi \in S_{\varepsilon,k}$.

From Lemma 2 by standard fixed point methods we obtain the following

Lemma 3

For any $\varepsilon \in (0,1)$ *there exists* $\overline{k} = \overline{k}(\varepsilon) > 0$ *such that for every* $k > \overline{k}$ *there exists a solution* $w^* \in S_{\varepsilon,k}$ *of the problem*

$$\begin{cases} -\Delta w = G(\cdot, k + w) & \text{in } \Omega \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{14}$$

The following result is the crucial step in the proof of the "if" part of Theorem 14.

Proposition 4

Let condition (12) *be satisfied. Then for any* $M > 0$ *there exists a stationary supersolution* $\overline{v}$ *of problem* (N) *such that* $\overline{v} \geq M$ *in* Ω.

Proof. (i) By (12) there holds

$$\varepsilon_0 := -\frac{\int_\Omega a(x)\,dx}{\int_\Omega |a(x)|\,dx} \in (0,1)\,.$$

Then for any $\varepsilon \in (0, \varepsilon_0)$ and any $k > 0$

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{m}} < \frac{1+\varepsilon}{1-\varepsilon} < \frac{1+\varepsilon_0}{1-\varepsilon_0} = \frac{\int_\Omega a_-(x)\,dx}{\int_\Omega |a_+(x)|\,dx}\,, \tag{15}$$

where $a_\pm := \frac{|a|\pm a}{2}$.
(ii) Let $\varepsilon \in (0, \varepsilon_0)$ and $k > \overline{k}(\varepsilon) > 0$. Set

$$\overline{v} := (k + w^*)^{\frac{1}{m}}\,, \tag{16}$$

where $w^* \in S_{\varepsilon,k}$ is the solution of problem (14), whose existence is ensured by Lemma 3.

It follows from (i) above that $\overline{v}$ is a supersolution of problem (NS). In fact, let us show that for any $\eta \in C^2(\overline{\Omega})$, $\eta \geq 0$ such that $\frac{\partial \eta}{\partial n} = 0$ on $\partial\Omega$ there holds

$$\int_\Omega \{\overline{v}^m \Delta\eta + a\overline{v}\eta\}\, dx \leq 0\,,$$

or equivalently

$$\int_\Omega \{\Delta\overline{v}^m + a\overline{v}\}\, \eta\, dx - \int_{\partial\Omega} \frac{\partial \overline{v}^m}{\partial n} \eta\, d\sigma \leq 0\,.$$

Observe that by Lemma 3

$$\frac{\partial \overline{v}^m}{\partial n} = \frac{\partial w^*}{\partial n} = 0 \quad \text{on } \partial B_k\,.$$

Hence we must show that

$$\int_\Omega \{\Delta\overline{v}^m + a\overline{v}\}\, \eta\, dx \leq 0\,. \tag{17}$$

By inequality (15)

$$\begin{aligned}
\Delta w^* + a\,(k+w^*)^{\frac{1}{m}} &= -G(\cdot, k+w^*) + a\,(k+w^*)^{\frac{1}{m}} = \\
&= \frac{1}{|\Omega|}\int_\Omega a\,(k+w^*)^{\frac{1}{m}}\,dx \le \\
&\le \frac{k^{\frac{1}{m}}}{|\Omega|}\left\{(1+\varepsilon)^{\frac{1}{m}}\int_\Omega a_+\,dx - (1-\varepsilon)^{\frac{1}{m}}\int_\Omega a_-\,dx\right\} \le 0\,.
\end{aligned}$$

Here we have used the following remark:

$$w^* \in S_{\varepsilon,k} \quad \Leftrightarrow \quad \|w^*\|_\infty < \varepsilon k \quad \Rightarrow \quad -\varepsilon_k \le w^* \le \varepsilon_k \text{ in } \Omega\,,$$

whence

$$a\,(k+w^*)^{\frac{1}{m}} \le [k(1+\varepsilon)]^{\frac{1}{m}}\,a_+ - [k(1-\varepsilon)]^{\frac{1}{m}}\,a_-\,.$$

Therefore, inequality (17) follows.

(*ii*) Since by construction $\overline{v}^m \ge k(1-\varepsilon)$ in Ω, by choosing $\varepsilon \in (0,\varepsilon_0)$ and $k > \overline{k}(\varepsilon) > 0$ so large that $k(1-\varepsilon) > M^m$ the result follows. □

Proof of Theorem 14 ("if" part)

Proof.

The proof is the same of Theorem 13, by using the same subsolution (11) and replacing the supersolution (10) by the supersolution (16) we have constructed above. □

Remark: The " only if" part of Theorem 14 follows from Theorem 10 for time periodic solutions, which will be proven below.

Remark: Concerning the multiplicity of nonnegative solutions, the situation is analogous to that of the Dirichlet case.

Let us summarize:

So far we have discussed:

- Statement of the problem and motivations
- Definitions and preliminary results
- Main results:
  - Existence of periodic solutions
  - The role of boundary conditions: Dirichlet versus Neumann
- Monotonicity methods
- Sketch of the proofs:
  - Construction of stationary sub- and supersolutions

To be discussed yet:

- Sketch of the proofs:
  - Construction of T-periodic sub- and supersolutions
- Multiplicity and support properties (overview)

The time periodic case: Constructing subsolutions

(Dirichlet or Neumann boundary conditions)

To prove existence of T-periodic solutions for problems (D) and (N) we shall make use as before of monotonicity methods. Again the proof relies on Proposition 1, and requires constructing for each problem an ordered couple of T-periodic sub- and supersolutions.
The construction is similar to that of stationary sub- and supersolutions, yet more complicated.

Recall that by assumption (see (5))

$$\Omega^+ := \{x \in \Omega \mid \int_0^T a(t,x)\,dx > 0\} \neq \emptyset\,, \quad \Omega^+ = \bigcup_{k \in \mathcal{M}} \Omega_k^+\,.$$

Let us begin by exhibiting, both for problem (D) and for problem (N), small T-periodic subsolutions.

Small T-periodic subsolutions: Let B_k be any open ball such that $\overline{B}_k \subset \Omega_k^+$ $(k \in M)$. Define

$$\underline{a}_k(t) := \min_{x \in \overline{B}_k} a(t, x) \qquad (t \in \mathbb{R}).$$

For any $x_0 \in B_k$ there holds

$$\int_0^T a(t, x_0)\, dx > 0,$$

whence by the uniformy continuity of the map a in $[0, T] \times \overline{\Omega}$

$$\int_0^T \underline{a}_k(t)\, dt > 0$$

(possibly in a smaller ball, denoted again B_k for simplicity). Consider the eigenvalue problem

$$\begin{cases} \Delta\xi_k + \lambda_k \xi_k = 0 & \text{in } B_k \\ \xi_k = 0 & \text{on } \partial B_k \\ \xi_k > 0 & \text{in } B_k, \end{cases}$$

with $\|\xi_k\|_\infty = 1$.

Set

$$\varepsilon_k := \frac{1}{\lambda_k T} \int_0^T \underline{a}_k(t)\, dt > 0\,,$$

$$\alpha_k(t) := \exp\left\{ \int_0^t \underline{a}_k(s)\, ds - \varepsilon_k \lambda_k t \right\} \qquad (t \in \mathbb{R})\,.$$

Observe that for any $t \in \mathbb{R}$

$$\alpha_k(t+T) = \exp\left\{ \int_0^{t+T} \underline{a}_k(s)\, ds - \varepsilon_k \lambda_k t - \int_0^T \underline{a}_k(s)\, ds \right\} = \alpha_k(t)\,,$$

thus α_k is T-periodic. Moreover,

$$\alpha_k'(t) = [\underline{a}_k(t) - \varepsilon_k \lambda_k]\, \alpha_k(t)\,.$$

We shall prove that the function

$$\underline{v}_{k,\rho}(t,\cdot) := \begin{cases} \alpha_k(t)\, (\rho \xi_k)^{\frac{1}{m}} & \text{if } (t,x) \in \mathbb{R} \times B_k \\ 0 & \text{if } (t,x) \in \mathbb{R} \times (\Omega \setminus B_k) \end{cases}$$

is a T-periodic subsolution of both problems (D) and (N), if $\rho > 0$ is sufficiently small.

In fact, let us show that for every $\tau \in (0, T]$ and any $\eta \in C^2(\overline{Q_\tau})$, $\eta \geq 0$ such that $\eta = 0$ (respectively, $\frac{\partial \eta}{\partial n} = 0$) in $[0, \tau] \times \partial\Omega$ there holds (by setting $\underline{v} \equiv \underline{v}_{k,\rho}$ for simplicity)

$$\int_\Omega \underline{v}(\tau, x)\eta(\tau, x)\, dx - \int_\Omega \underline{v}(0, x)\eta(0, x)\, dx \leq$$

$$\leq \int\int_{Q_\tau} \{\underline{v}\partial_t \eta + \underline{v}^m \Delta\eta + a\underline{v}\eta\}\, dtdx\,.$$

Integrating by parts we get the equivalent inequality

$$\int\int_{[0,\tau]\times B_k} \{\partial_t \underline{v} - \Delta \underline{v}^m - a\underline{v}\}\, \eta\, dtdx + \int_0^\tau dt \int_{\partial B_k} \frac{\partial \underline{v}^m}{\partial n} \eta\, d\sigma \leq 0\,. \quad (18)$$

Observe that

$$\partial_t \underline{v} = \alpha_k'(t)\, (\rho\xi_k)^{\frac{1}{m}} = [\underline{a}_k(t) - \varepsilon_k \lambda_k]\, \alpha_k(t)\, (\rho\xi_k)^{\frac{1}{m}} = [\underline{a}_k(t) - \varepsilon_k\lambda_k]\, \underline{v}\,,$$

$$\Delta \underline{v}^m = \alpha_k(t)^m \rho \Delta\xi_k = -\lambda_k \alpha_k(t)^m \rho\xi_k = -\lambda_k \underline{v}^m \quad \text{in } B_k\,,$$

$$\frac{\partial \underline{v}^m}{\partial n} = \alpha_k(t)^m \rho \frac{\partial \xi_k}{\partial n} < 0 \quad \text{on } \partial B_k\,.$$

Then

$$\int\int_{[0,\tau]\times B_k} \{\partial_t \underline{v} - \Delta \underline{v}^m - a\underline{v}\}\,\eta\, dtdx + \int_0^\tau dt \int_{\partial B_k} \frac{\partial \underline{v}^m}{\partial n}\eta\, d\sigma \leq$$

$$\leq \int\int_{[0,\tau]\times B_k} \{\partial_t \underline{v} - \Delta \underline{v}^m - a\underline{v}\}\,\eta\, dtdx =$$

$$= \int\int_{[0,\tau]\times B_k} \{[\underline{a}_k(t) - \varepsilon_k \lambda_k]\,\underline{v} + \lambda_k \underline{v}^m - a\underline{v}\}\;\eta\, dx =$$

$$= \int\int_{[0,\tau]\times B_k} \left[\underbrace{\underline{a}_k(t) - a(t,x)}_{\leq 0} + \lambda_k\left(\underline{v}^{m-1} - \varepsilon_k\right)\right] \underline{v}\eta\, dx \leq$$

$$\leq \lambda_k \int\int_{[0,\tau]\times B_k} \left(\underline{v}^{m-1} - \varepsilon_k\right)\, \underline{v}\eta\, dx < 0\,,$$

$$\text{if} \quad \rho < \frac{\varepsilon_k^{\frac{m}{m-1}}}{\|\alpha_k\|_\infty^m}, \text{ which implies } \underline{v}^{m-1} < \varepsilon_k\,.$$

This proves (18), thus the claim follows.

The time periodic case: Existence of solutions

(Dirichlet boundary conditions)

Large T-periodic supersolutions (Dirichlet case): It suffices to make use of the stationary, thus T-periodic supersolutions constructed before for the Dirichlet case.

Now we can prove Theorem 9: *"Let the set*

$$\Omega^+ := \{x \in \Omega \mid \int_0^T a(t,x)\,dx > 0\}$$

be nonempty. Then there exists a nontrivial, nonnegative T-periodic solution of problem (D)."

Proof. We have constructed both "small" nontrivial T-periodic subsolutions and "large" T-periodic supersolutions of problem (D), and we can choose their parameters to obtain an ordered couple. Then by Theorem 12 T-periodic solutions of problem (D) exist. □

The time periodic case: Existence of solutions
(Neumann boundary conditions)

Now we want to prove Theorem 10: *"Let the set Ω^+ be nonempty, and*

$$\int\int_{Q_T} a(t,x)\,dtdx < 0\,. \tag{19}$$

Then there exists a nontrivial, nonnegative T-periodic solution of problem (N)."

We have already constructed "small" nontrivial T-periodic subsolutions of problem (N). To prove Theorem 10, it suffices to construct "large" T-periodic supersolutions of this problem. This will be made by a refinement of the construction used for stationary solutions of (N).

Remark: Condition (12), which was used to construct stationary supersolutions of (N), is *stronger* than condition (19). Therefore, we cannot use stationary supersolutions as in the Dirichlet case.

Large T-periodic supersolutions (Neumann case): For any $\varepsilon \in (0,1)$, $\theta > 0$ and $k_0 > 0$ define

$$c(t) \equiv c_{\varepsilon,\theta}(t) := \frac{1}{|\Omega|}\{(1+\varepsilon)\int_\Omega a$$

$$c($$

As for the stationary case, there holds

Lemma 5

For any $\varepsilon \in (0,1)$ *and* $\theta > 0$ *there exists* $\overline{k} = \overline{k}(\varepsilon, \theta) > 0$ *such that for every* $k_0 > \overline{k}$ *and for any* $t \in [0, T]$ *there exists a solution* $w^*(t, \cdot) \in S_{\varepsilon, k(t)}$ *of problem* (20).

Now the proof of Theorem 14 follows from the existence of "small" T-periodic subsolutions of problem (N) (already proven) and from the following proposition.

Proposition 6

Let condition (19) *be satisfied. Then for any* $M > 0$ *there exists a* T*-periodic supersolution* $\overline{v}$ *of problem* (N) *such that* $\overline{v} \geq M$ *in* $\overline{Q}_T$*.*

Hint of the proof. By (19) there holds

$$\varepsilon_0 := -\frac{\iint_{Q_T} a(t,x)\,dtdx}{\iint_{Q_T} |a(t,x)|\,dtdx} \in (0,1)\,,$$

whence

$$\theta_0 := \frac{2\varepsilon_0(1-\varepsilon_0)}{2+\varepsilon_0(1-\varepsilon_0)}\,\frac{1}{|\Omega|T}\int\int_{Q_T} a_-(t,x)\,dtdx > 0\,.$$

Let $\varepsilon \in (0,\varepsilon_0)$, $\theta \in (0,\theta_0)$ and $k_0 > \overline{k}(\varepsilon,\theta) > 0$. Set

$$\overline{v}(t,x) := [k(t) + w^*(x)]^{\frac{1}{m}} \qquad ((t,x) \in \overline{Q}_T)\,.,$$

where $w^* \in S_{\varepsilon,k(t)}$ is the solution of problem (20), whose existence is ensured by Lemma 5. It can be proven that $\overline{v}$ is the supersolution we seek.

Necessity of condition (6):

Let us finally prove Theorem 10 (which also entails the "only if" part of Theorem 14): *"Let there exist a T-periodic solution of problem (N), positive in $\overline{Q}_T$. Then*

$$\int\int_{Q_T} a(t,x)\,dtdx \le 0\,, \tag{21}$$

and equality holds if and only if a does not depend on x."

Proof. A T-periodic solution u of problem (N), positive in $\overline{Q}_T$, is a classical solution. Dividing by u the equation $\partial_t u = \Delta u^m + au$ and integrating in Q_T gives

$$\int\int_{Q_T} \left(\frac{\Delta u^m}{u} + a\right)\,dtdx = \int\int_{Q_T} \partial_t[\log u]\,dtdx = \int_\Omega \log\frac{u(T,x)}{u(0,x)}\,dx = 0$$

$$\Rightarrow \int\int_{Q_T} a\,dtdx = -\int\int_{Q_T} \frac{\Delta u^m}{u}\,dtdx = -m\int\int_{Q_T} u^{m-3}|\nabla u|^2\,dtdx \le 0\,.$$

If " = " holds in (21), then $u = u(t) \Rightarrow \partial_t u = au \Rightarrow a = a(t)$. $\square$

Multiplicity and support properties

Let us denote by $\mathcal{M} \subseteq \mathbb{N}$ any family of integer. We already used the fact that the set Ω^+ is a countable union of maximal connected components Ω_k^+, namely

$$\Omega^+ = \bigcup_{k\in\mathcal{M}} \Omega_k^+ , \quad \Omega_k^+ \cap \Omega_l^+ = \emptyset \;\text{ if } k \neq l \qquad (k, l \in \mathcal{M}) \tag{22}$$

Concerning support properties of T-periodic solutions of problems (D) and (N), the following holds.

Theorem 15

Let u be a nontrivial, nonnegative T-periodic solution either of problem (D), or of problem (N). Then:

(i) the support of $u(t,\cdot)$ does not depend on t;

(ii) if the set Ω^+ is nonempty, either $u > 0$, or $u \equiv 0$ in $(0, T] \times \Omega_k^+$ for any $k \in \mathcal{M}$.

In view of Theorem 15, the existence Theorem 9 can be refined as follows; a similar result holds for solutions of problem (N) (see Theorem 10).

Theorem 16

Let the set Ω^+ *be nonempty. Then for any nonempty subset* $\mathrm{I} \,\square\, \mathrm{M}$ *there exists a nonnegative* T*-periodic solution of problem* (D)*, which is positive in* $(0, T] \times \Omega_k^+$ *for any* $k \,\square\, \mathrm{I}$*.*

In particular, Theorem 19 implies the existence of a T-periodic solution of problem (D) which is positive in $(0, T] \times \Omega^+$ (this generalizes Theorem 13 concerning stationary solutions).

Remark: A T-periodic solution positive in $(0, T] \times \Omega_k^+$ can vanish identically in $(0, T] \times \Omega_l^+$ for some $l \neq k$. Hence *the multiplicity of* T*-periodic solutions either of problem* (D)*, or of problem* (N) *can depend on the number of connected components of* Ω^+*.*

Multiplicity and support properties: The autonomous case

For any $\mathrm{I} \subseteq \mathrm{M}$ set $\Omega^+_{\mathcal{I}} := \cup_{k\in\mathcal{I}} \Omega^+_k$. To describe in the autonomous case stationary solutions of problems (D) and (N), the following definition is useful.

Definition 17

For any $\mathrm{I} \subseteq \mathrm{M}$ we denote by $\mathrm{S}_{\mathcal{I}}$ the set of stationary solutions of (D) which are positive in $\Omega^+_{\mathcal{I}}$. We also set:

$$\mathrm{N}_{\mathcal{I}} := \{v \in \mathrm{S}_{\mathcal{I}} \mid v \equiv 0 \text{ in } \Omega \setminus \Omega^+_{\mathcal{I}}\} .$$

Theorem 18

(i) *Every stationary solution of problem* (D) *belongs to some set* $\mathrm{N}_{\mathcal{I}}$.
(ii) *Let* $\mathrm{I} \subseteq \mathrm{M}$ *be finite. Then there exists at most one stationary solution* $v \in \mathrm{N}_{\mathcal{I}}$ *of problem* (D).
(iii) *For any* $\mathrm{I} \subseteq \mathrm{M}$ *the set* $\mathrm{S}_{\mathcal{I}}$ *is nonempty; moreover, it has a minimal and a maximal element.*

As a consequence of Theorem 18 we have the following result.

Theorem 19

(i) Let the set Ω^+ be nonempty. Then there exists a nontrivial stationary solution of problem (D).
(ii) There exists a unique stationary solution of problem (D) which is positive in Ω^+. This solution is the maximal element of the set $S_{\mathcal{I}}$ for any I □ M *.*
(iii) Every other stationary solution of problem (D) identically vanishes in some subset Ω_k^+.

A similar situation holds true for stationary solutions of problem (*N*) in the autonomous case, and for *T*-periodic solutions of problems (*D*) and (*N*) in the general case.