Two topics in the theory of reaction-diffusion equations

Lecture notes from Program on Dynamics of Periodic Equations,
Center for PDE, ECNU, mid March-May, 2015

Alberto Tesei
University of Rome, Italy.tesei@mat.uniroma1.it

April 22,23,28,30, May 5

The Center receives funding from

RECRUITMENT
PROGRAM OF GLOBAL EXPERTS

Cente for PDE
500 Dongchuan Road
Administration Building 12th floor
Minhang Campus, East China Normal University
Shanghai, 200241, China
Email: admin@cpde.ecnu.edu.cn

Forward-backward parabolic equations, 2

Alberto Tesei

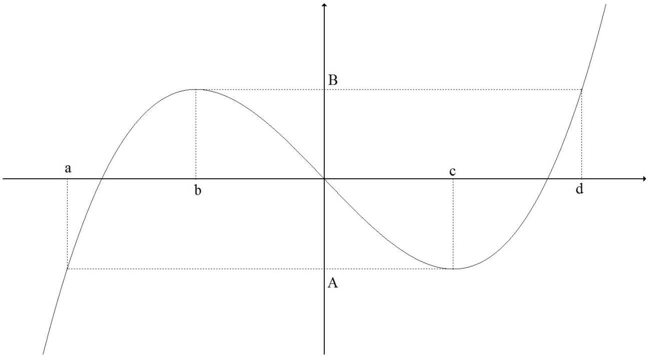
University of Rome "La Sapienza"
Istituto per le Applicazioni del Calcolo "M. Picone", CNR

Center for PDE, East China Normal University
Shanghai - April 28, 2015

Plan of the lecture

- Cubic ϕ with Sobolev regularization [Novick-Cohen & Pego]
- Singular limit as $\varepsilon \to 0$ [Plotnikov]: A few general remarks
- Two-phase solutions
- Hyperbolic first order conservation laws: A reminder
- Singular limit as $\varepsilon \to 0$:
  - Entropy conditions and interfaces [Evans & Portilheiro]
  - Existence and uniqueness of two phase solutions

In this lecture: ϕ of cubic type



Consider the problem

$$(C) \qquad \begin{cases} u_t = \Delta[\phi(u)] & \text{in } \Omega \times (0,T] =: Q_T \\ u = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 \in L^\infty(\Omega) & \text{in } \Omega \times \{0\}\,. \end{cases}$$

Main idea ([Plotnikov, 1994]): To define *weak entropy solutions* of problem (C) by the Sobolev regularization

$$(C_\epsilon) \qquad \begin{cases} u_t = \Delta v & \text{in } Q_T \\ v = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 \in L^\infty(\Omega) & \text{in } \Omega \times \{0\}\,, \end{cases}$$

letting $\epsilon \to 0$ in the *chemical potential*

$$v := \phi(u) + \epsilon u_t\,.$$

Remark: For every $\epsilon > 0$ there exists a unique strong solution (u_ϵ, v_ϵ) of the regularized problem (C_ϵ). [Novick-Cohen & Pego]

Sobolev regularization

Novick-Cohen & Pego studied the Neumann IBVP

$$\begin{cases} u_t = \Delta v & \text{in } Q_T \\ \frac{\partial v}{\partial n} = 0 & \text{in } \partial\Omega \times (0, T] \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{1}$$

where $v := \phi(u) + \epsilon u_t$, $\Omega \subseteq \mathbb{R}^n$ bounded, $\partial\Omega$ smooth, $T > 0$.

Definition 1

Let $u_0 \in L^\infty(\Omega)$. By a *solution* to problem (1) in Q_T we mean any couple $u \in C^1([0, T]; L^\infty(\Omega))$, $v \in C([0, T]; L^\infty(\Omega) \cap W^{2,p}(\Omega))$ with $p > n$, $\Delta v \in C([0, T]; L^\infty(\Omega))$ which satisfies (1) in the strong sense.

Existence and uniqueness:

Theorem 1

For any $\epsilon > 0$ there exists a unique solution (u_ϵ, v_ϵ) of problem (1).

Proof. (i) By ODE theory in Banach spaces, there exist $T_\epsilon > 0$ and a unique $u_\epsilon \in C^1([0, T_\epsilon); L^\infty(\Omega))$ satisfying

$$\begin{cases} u_t = \Big(I - \epsilon\Delta\Big)^{-1} \Delta\phi(u) & \text{in } Q_{T_\epsilon} \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases}$$

In view of the a priori estimate (7) proven below, this solution is global.
(ii) By standard elliptic theory, for any $g \in L^\infty(\Omega)$ there exists a unique solution $w \in W^{2,p}(\Omega)$ $(p \in (1, \infty))$ of the problem

$$\begin{cases} (I - \epsilon\Delta)\, w = g & \text{in } \Omega \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Taking $g = \phi(u_\epsilon(\cdot, t))$ $(t \in [0, T])$ proves the result. □

A priori estimates

For any $g \in C^1(\mathbb{R})$, $g' \geq 0$ set

$$G(z) := \int_0^z g(\phi(s))ds + c\,. \tag{2}$$

For any solution (u_ϵ, v_ϵ) of (1), a formal calculation gives:

$$\begin{aligned}
[G(u_\epsilon)]_t &= g(\phi(u_\epsilon))u_{\epsilon t} = g(\phi(u_\epsilon))\Delta v_\epsilon = \\
&= g(v_\epsilon)\Delta v_\epsilon + \Big[g(\phi(u_\epsilon)) - g(v_\epsilon)\Big]\Delta v_\epsilon = \\
&= div\Big[g(v_\epsilon)\nabla v_\epsilon\Big] - g'(v_\epsilon)|\nabla v_\epsilon|^2 + \\
&+ \underbrace{\Big[g(\phi(u_\epsilon)) - g(v_\epsilon)\Big]\frac{v_\epsilon - \phi(u_\epsilon)}{\epsilon}}_{\leq 0}
\end{aligned} \tag{3}$$

Integrating in Ω we obtain:

$$\frac{d}{dt}\int_\Omega G(u_\epsilon(x,t))\,dx \leq 0\,. \tag{4}$$

From (3) we also obtain:

$$\iint_{Q_T} \left\{ G(u_\epsilon)\zeta_t - g(v_\epsilon)\nabla v_\epsilon \cdot \nabla\zeta - g'(v_\epsilon)|\nabla v_\epsilon|^2 \zeta \right\} dxdt \geq 0 \qquad (5)$$

for any $\zeta \in C_0^\infty(Q_T)$, $\zeta \geq 0$.

Remark: Interesting analogy between (5) and the *entropy inequality* for the *viscous conservation law*:

$$u_t + [f(u)]_x = \epsilon u_{xx} \ \text{ in } \mathbb{R} \times (0, T] =: S_T \,.$$

This is:

$$\iint_{S_T} \left\{ E(u_\epsilon)[\zeta_t + \epsilon\zeta_{xx}] + F(u_\epsilon)\zeta_x \right\} dxdt \geq 0$$

for any ζ as above and any couple *entropy-flux*

$$(E, F) \Leftrightarrow E, F \in C^1(\mathbb{R}), \ \ F' = f'E' \,, \ \ E \text{ convex} \,.$$

By analogy, inequality (5) is called *viscous entropy inequality* for the Sobolev equation.

Inequality (4) is also crucial to prove the existence of *positively invariant regions* for problem (1). In fact, we have:

Proposition 2

Assume that

$$\phi(u_1) \leq \phi(u) \leq \phi(u_2) \quad \text{for any} \quad u \in [u_1, u_2]; \tag{6}$$

moreover, let $u_0(x) \in [u_1, u_2]$ *for any* $x \in \Omega$. *Then* $u_\epsilon(x,t) \in [u_1, u_2]$ *for any* $(x,t) \in Q_{T_\epsilon}$.

Proof. It is not restrictive (possibly changing the definition of ϕ out of $[u_1, u_2]$) to assume $\phi(u) < \phi(u_1)$ if $u < u_1$, $\phi(u) > \phi(u_2)$ if $u > u_2$. Fix $g \in C^1(\mathbb{R})$, $g' \geq 0$ such that $g \equiv 0$ in $[\phi(u_1), \phi(u_2)]$, $g(z) < 0$ if $z < \phi(u_1)$, $g(z) > 0$ if $z > \phi(u_2)$; moreover, choose the constant c in (2) such that $G(u) = \int_{u_1}^{u} g(\phi(s))ds$. Plainly, this implies $G \equiv 0$ in $[u_1, u_2]$, $G > 0$ in $(-\infty, u_1) \cup (u_2, \infty)$. Then by inequality (4) the conclusion follows.

Proposition 3

Let (u_ϵ, v_ϵ) be a solution of problem (1). Then there exists $C_1 > 0$ such that for any $\epsilon > 0$:

$$||u_\epsilon||_{L^\infty(Q_T)} \leq C_1 ||u_0||_\infty . \tag{7}$$

Proposition 4

Let (u_ϵ, v_ϵ) be a solution of problem (1). Then there exists $C_2 > 0$ such that for any $\epsilon > 0$:

$$\|v_\epsilon\|_{L^2((0,T);H^1(\Omega))} + \|\sqrt{\epsilon} u_{\epsilon t}\|_{L^2(Q_T)} \leq C_2 . \tag{8}$$

Proof. Choosing $g(z) = z$ in (3) and using (7) gives the result:

$$[G(u_\epsilon)]_t = v_\epsilon \Delta v_\epsilon - \frac{|v_\epsilon - \phi(u_\epsilon)|^2}{\epsilon} = v_\epsilon \Delta v_\epsilon - \epsilon |u_{\epsilon t}|^2$$

$$\Rightarrow \quad \iint_{Q_T} \left\{ |\nabla v_\epsilon|^2 + \epsilon |u_{\epsilon t}|^2 \right\} dxdt \leq \int_\Omega \left\{ G(u_0) - G(u_\epsilon(T)) \right\} dx .$$

Proposition 5

Let (u_ϵ, v_ϵ) *be a solution of problem (1). Then there exists* $C_3 > 0$ *such that for any* $\epsilon > 0$*:*

$$\|v_\epsilon\|_{L^\infty(Q_T)} \leq C_3 \,. \tag{9}$$

Proof. Since the function ϕ is continuous, the map $t \to ||\phi(u_\epsilon)(\cdot, t)||_\infty$ is bounded in $[0, \infty)$. Moreover, $\|u_{\epsilon t}\|_{L^\infty(Q_T)} < \infty$ since $u_\epsilon \in C^1([0, T]; L^\infty(\Omega))$. Since by definition $v_\epsilon = \phi(u_\epsilon) + \epsilon u_{\epsilon t}$, the result follows.

Singular limit: Sending $\varepsilon \to 0$

Theorem 2 (Plotnikov)

There exist $u, v, \lambda_0, \lambda_1, \lambda_2 \in L^\infty(Q_T)$, $v \in C((0,T]; H^1(\Omega))$ *such that:*

(a) $\sum_{i=0}^{2} \lambda_i = 1$, $\lambda_i \geq 0$, $\lambda_1 = 1$ *if* $v < A$, $\lambda_2 = 1$ *if* $v > B$, *and*

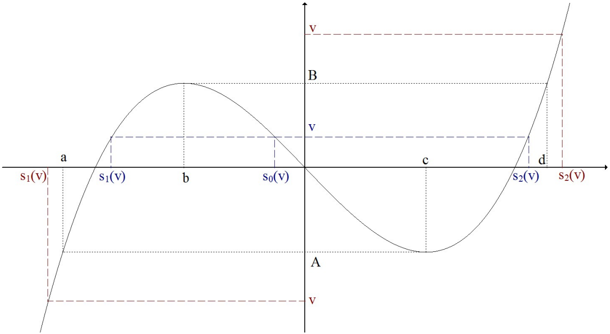
$$u = \sum_{i=0}^{2} \lambda_i s_i(v)\,; \tag{10}$$

(b) *for any* $\zeta \in C^1(\overline{Q}_T)$, $\zeta(\cdot, T) = 0$ *in* $\overline{\Omega}$

$$\iint_{Q_T} (u\zeta_t - \nabla v \cdot \nabla \zeta)\, dxdt + \int_\Omega u_0 \zeta(x,0)\, dx = 0\,;$$

(c) *for any* $\zeta, g \in C^1(\mathbb{R})$, $g' \geq 0$

$$\iint_{Q_T} \Big\{ \sum_{i=0}^{2} \lambda_i G(s_i(v))\zeta_t - g(v)\nabla v \cdot \nabla \zeta - g'(v)|\nabla v|^2 \zeta \Big\} dxdt \geq 0\,. \tag{11}$$


v
B
v
a
c
d
s1(v)
s1(v)
b
s0(v)
s2(v)
s2(v)
A
v

In this way we get a *Young measure-valued solution* of problem (C). In fact, suppose we define the *probability measure*

$$\nu_{x,t}(\tau) := \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - s_i(v(x,t)))$$

for any $(x,t) \in Q_T$, $\tau \in \mathbb{R}$. Then by equality (10)

$$\int_{\mathbb{R}} \tau d\nu_{x,t}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t) s_i(v(x,t)) = u(x,t)\,.$$

Also,

$$\int_{\mathbb{R}} \phi(\tau) d\nu_{x,t}(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\phi(s_i(v(x,t))) = v(x,t)\,.$$

Hence we have:

$$\iint_{Q_T} \Big\{ \zeta_t \underbrace{\int_{\mathbb{R}} \tau d\nu_{x,t}(\tau)}_{=u(x,t)} - \nabla\zeta \cdot \nabla \underbrace{\int_{\mathbb{R}} \phi(\tau) d\nu_{x,t}(\tau)}_{=v(x,t)} \Big\} dxdt + \int_{\Omega} u_0(x)\zeta(x,0)dx = 0\,.$$

INTERPRETATION: The function u takes the *fraction* λ_i of its value at (x,t) on the branch $s_i(v)$ of the graph of $v = \phi(u)$. The functions $\lambda_0, \lambda_1, \lambda_2$ are called *phase fractions*. In the particular case $\lambda_1 = 1$, $\lambda_0 = \lambda_2 = 0$ *a.e.* in Q_T, $u = s_1(v)$ is a weak solution of problem (C) in the usual sense: in this case we are on the stable branch $u = s_1(v)$ with $v < A$. In general, u is a *superposition of different phases* (see (10)).

To summarize: *a*) Theorem 2 proves the existence of Young measure-valued solutions of problem (C).
b) Inequality (11) is obtained as the limit of the viscous entropy inequality (5), thus it is called *entropy inequality* of problem (C) (recall that g is arbitrary!). Therefore, the above solutions are called *entropy solutions* of (C).

Remark: No result is known about uniqueness of the above solutions of (C). Namely, in this theory there is no counterpart of the Kružkov uniqueness theorem for first order hyperbolic conservation laws.

Cubic versus Perona-Malik :

The main tool of the above analysis is the uniform L^∞-estimate of $\{u_\epsilon\}$. This follows from the existence of *positively invariant regions* [Novick-Cohen & Pego] :

Assume

$$\phi(u_1) \leq \phi(u) \leq \phi(u_2) \text{ for any } u \in [u_1, u_2]\,. \qquad (12)$$

Then, if $u_0(x) \in [u_1, u_2]$ for every $x \in \Omega$, there holds $u_\epsilon(x,t) \in [u_1, u_2]$ for every $(x,t) \in Q$.

- If ϕ is of cubic type, every $[u_1, u_2]$ sufficiently large satisfies condition (12). Hence we get the L^∞-estimate of $\{u_\epsilon\}$, uniform with respect to ϵ.
- If ϕ is of Perona-Malik type, only the intervals $[u_1, u_2] \subseteq [-\alpha, \alpha]$ satisfy condition (12). Hence an L^∞-estimate of $\{u_\epsilon\}$ only holds *if* $\{u_\epsilon\}$ *takes values in the stable phase* (trivial case).

Remark: If ϕ is of Perona-Malik type, the half-line $[0, +\infty)$ is positively invariant for solutions of the regularized problem.

A case study: Two-phase solutions

Following [Evans-Portilheiro], assume that

$$(H) \qquad \lambda_0 = 0 \;\; a.e. \text{ in } Q_T, \quad \lambda_i = 1 \; a.e. \text{ in } V_i \qquad (i = 1, 2) \;,$$

where

$$\overline{Q}_T = \overline{V}_1 \cup \overline{V}_2 \,, \quad V_1 \cap V_2 = \emptyset$$

and

$$\gamma := \overline{V}_1 \cap \overline{V}_2$$

is a smooth surface.

Definition 3

If λ_0, λ_1 and λ_2 satisfy assumption (H), a solution of problem (C) given by Theorem 2 is called a *two-phase solution*.

Let $n = 1$ for simplicity. Then

$$\gamma = \{(\xi(t), t) \mid t \in [0, T]\}, \quad \xi \in C^1(0, T),$$

and the solutions of problem (C) given by Theorem 2 satisfy the following:

$$u = s_i(v) \ \ a.e. \ in \ V_i \quad (i = 1, 2);$$

$$\iint_{Q_T} (u\zeta_t - v_x\zeta_x)\, dxdt = \sum_{i=1}^{2} \iint_{V_i} \left\{ s_i(v)\zeta_t - v_x\zeta_x \right\} dxdt = 0$$

for any $\zeta \in C_0^\infty(Q_T)$;

$$\iint_{Q_T} \left\{ G(u)\zeta_t - g(v)v_x\zeta_x - g'(v)|v_x|^2\zeta \right\} dxdt =$$

$$= \sum_{i=1}^{2} \iint_{V_i} \left\{ G(s_i(v))\zeta_t - g(v)v_x\zeta_x - g'(v)|v_x|^2\zeta \right\} dxdt \geq 0$$

for any $g \in C^1(\mathbb{R})$, $g' \geq 0$, $\zeta \in C_0^\infty(Q_T)$, $\zeta \geq 0$.

Theorem 4

Let $u, v, \lambda_0, \lambda_1, \lambda_2$ be as in Theorem 2, with $\lambda_0, \lambda_1, \lambda_2$ satisfying assumption (H). Then u, v have the following properties:
(i) *u, v are classical solutions of*

$$u_t = v_{xx} \quad \text{in } V_i \qquad (i = 1, 2).$$

Besides, v is continuous in Q_T;
(ii) *there holds the* Rankine-Hugoniot condition:

$$\xi' = -\frac{[v_x]}{[u]} \quad \textit{a.e. on } \gamma\,;$$

(iii) *there holds the* entropy condition:

$$\xi' \, [G(u)] \geq -g(v)[v_x] \quad \textit{a.e. on } \gamma\,.$$

Here $[h] := h^+(\xi(t), t) - h^-(\xi(t), t)$ denotes the jump across γ of any piecewise continuous function h.

First order hyperbolic conservation laws

Let us recall some results concerning the first order equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[f(u)] = 0 \qquad \text{in } \mathbb{R} \times (0, \infty) =: S \tag{13}$$

with the initial condition

$$u = u_0 \qquad \text{in } \mathbb{R} \times \{0\}. \tag{14}$$

A prototype of equation (13) is the *Burgers equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,, \quad \text{where } \; f(u) = \frac{u^2}{2}\,.$$

Remark 1: No global existence of classical solutions, even if $f \in C^2(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$, depending on the sign of f', u_0'.

Example 5

$$u(x,t) = \frac{x}{1+t} \quad (t \in (0,\infty))\,; \qquad u(x,t) = -\frac{x}{1-t} \quad (t \in (0,1))\,.$$

Therefore, *weak solutions* are introduced.

Definition 6

Let $u_0 \in L^1_{\text{loc}}(\mathbb{R})$. A function $u : \overline{S} \to \mathbb{R}$ is called *weak solution of problem* (13)-(14) if $u \in L^1_{\text{loc}}(S)$, $f(u) \in L^1_{\text{loc}}(S)$ and

$$\iint_S \{u\zeta_t + f(u)\zeta_x\}\, dxdt + \int_{\mathbb{R}} u_0\zeta(x,0)\, dx = 0 \qquad \forall \zeta \in C^1_0(\overline{S}). \quad (15)$$

Remark 1: No uniqueness of weak solutions. Counterexamples to uniqueness are easily found by considering *piecewise smooth solutions*.

Notations: Let $\Omega \subseteq S$ and $\Omega_\pm \subset \Omega$ be open. Suppose that

$$\Omega = \Omega_- \cup \gamma_0 \cup \Omega_+,$$

where

$$\gamma_0 \equiv \{(\xi(t), t) \mid t \in I\}$$

is the graph of a C^1 function $\xi = \xi(t)$.

The *jump of discontinuity* of u and of the *flux* $f(u)$ across γ_0 are

$$[u] \equiv [u](\xi(t), t) := u_+(\xi(t), t) - u_-(\xi(t), t)\,,$$

$$[f(u)] \equiv [f(u)](\xi(t), t) := f(u_+(\xi(t), t)) - f(u_-(\xi(t), t)) \qquad (t \in I)\,.$$

The following theorem characterizes weak solutions of class $C^1(\overline{\Omega}_\pm)$, possibly discontinuous across γ_0 (like in the *Riemann problem:)*

Theorem 7

Let $u \in C^1(\overline{\Omega}_\pm)$. Equivalent statements:
(a) u is a weak solution of equation (13) *in* Ω;

$$(b) \quad \begin{cases} (i)\ u \text{ is a classical solution of equation (13) in } \Omega_\pm\,; \\ (ii)\ \text{the Rankine-Hugoniot equation is satisfied:} \\ (RH) \qquad \xi' = \dfrac{[f(u)]}{[u]} \quad \text{on } \gamma_0\,. \end{cases}$$

Example 8

Rarefaction wave:

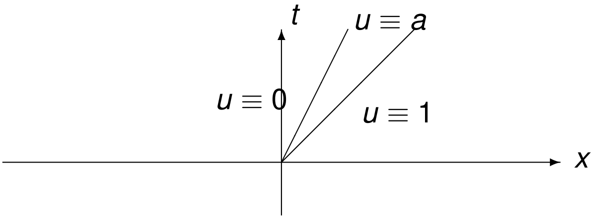
$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 & \text{in } S \\ u = \chi_{(0,\infty)} & \text{in } \mathbb{R} \times \{0\}. \end{cases} \quad (16)$$

For any $a \in (0,1)$ the function

$$u(x,t) := \begin{cases} 0 & \text{se } x \leq \frac{a}{2}t, \geq 0 \\ a & \text{se } \frac{a}{2}t \leq x \leq \frac{1+a}{2} \quad t \geq 0 \\ 1 & \text{se } \frac{1+a}{2}t < x \quad t \geq 0 \end{cases}$$

is a weak solution of problem (16).

Regularization: Adding a viscosity term



Therefore, *the Cauchy problem* (13)-(14) *is ill-posed*. In fact,
- no global existence of classical solutions;
- no uniqueness of weak solutions.

To regularize it, we consider the associated *viscous problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}[f(u)] = \epsilon \dfrac{\partial^2 u}{\partial x^2} & \text{in } S \qquad (\epsilon > 0) \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}. \end{cases} \tag{17}$$

Theorem 9

Let $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$*.*
(i) *There exists a unique classical solution* $u_\epsilon \in C^2(S) \cap C(\overline{S})$ *of problem* (17) *in* S*. Moreover,* $u_\epsilon \in L^\infty(S)$*, and*

$$\|u_\epsilon\|_{L^\infty(S)} \leq \|u_0\|_{L^\infty(\mathbb{R})}. \tag{18}$$

(ii) *Let there exist the limit*

$$\lim_{\epsilon \to 0^+} u_\epsilon =: u \qquad a.e.\ in\ S\,. \tag{19}$$

Then u *is a weak solution of the Cauchy problem* (13)-(14)*.*

Definition 10

The solution of problem (13)-(14) given by Theorem 9-(ii) is called *viscosity solution*. The limit in (19) is called *vanishing viscosity limit*.

Proof. For any $\zeta \in C_0^1(\overline{S})$ we have the weak formulation of the regularized problem (17):

$$0 = \iint_S \left\{ \frac{\partial u_\epsilon}{\partial t} + \frac{\partial}{\partial x}\left[f(u_\epsilon)\right] - \epsilon \frac{\partial^2 u_\epsilon}{\partial x^2} \right\} \zeta \, dxdt$$
$$= - \iint_S \left\{ u_\epsilon \zeta_t + f(u_\epsilon)\zeta_x + \epsilon u_\epsilon \frac{\partial^2 \zeta}{\partial x^2} \right\} dxdt - \int_{\mathbb{R}} u_0 \zeta(x,0)\, dx.$$

Letting $\epsilon \to 0^+$, by estimate (18) (uniform with respect to ϵ!), the Dominated Convergence Theorem, and the limit in (19) we get

$$0 = \iint_S \{u\zeta_t + f(u)\zeta_x\} \, dxdt + \int_{\mathbb{R}} u_0 \zeta(x,0)\, dx \,.$$

$\square$

Entropy solutions

Definition 11

A couple (E, F) of functions belonging to $C^1(\mathbb{R})$ is called an *entropy-entropy flux* (or simply *entropy-flux*) *couple* for equation (13) if:
- E is convex;
- there holds $F'(u) = f'(u)E'(u)$ for any $u \in \mathbb{R}$.

Diamo ora l'importante

Definition 12

A weak solution of the Cauchy problem (13)-(14) is called *entropy solution*, if for any entropy-flux couple (E, F) and for any $\zeta \in C_0^1(\overline{S})$, $\zeta \geq 0$ it satisfies the *entropy inequality:*

$$\iint_S \{E(u)\zeta_t + F(u)\zeta_x\}\, dxdt + \int_{\mathbb{R}} E(u_0)\zeta(x,0)\, dx \geq 0 . \qquad (20)$$

Entropy solutions are a class of well-posedness:

Existence of entropy solutions:

Theorem 13

Let u be a viscosity solution of problem (13)-(14)*. Then u is an entropy solution.*

Uniqueness of entropy solutions is given by the famous:

Theorem 14 (Kružkov)

Let $u_0 \in L^\infty(\mathbb{R})$. Then there exists at most one entropy solution of problem (13)-(14)*. Moreover, $u \in C\left([0,\infty); L^1_{loc}(\mathbb{R})\right) \cap L^\infty(S)$, and there holds*

$$\|u\|_{L^\infty(S)} = \|u_0\|_{L^\infty(\mathbb{R})}.$$

Hint of the proof of Theorem 13. Let $E \in C^2(\mathbb{R})$. By definition, u is an $a.e.$-limit of solutions u_ϵ of the regularized problem (17), thus

$$\frac{\partial u_\epsilon}{\partial t} + \frac{\partial}{\partial x}[f(u_\epsilon)] - \epsilon \frac{\partial^2 u_\epsilon}{\partial x^2} = 0 \quad \text{in } S \qquad (\epsilon > 0)$$

Since by definition $F' = f'E'$ in $\mathbb{R}$, multiplying by $E'(u_\epsilon)$ gives

$$\frac{\partial}{\partial t}[E(u_\epsilon)] + \frac{\partial}{\partial x}[F(u_\epsilon)] - \epsilon E'(u_\epsilon)\frac{\partial^2 u_\epsilon}{\partial x^2} = 0 \qquad \text{in } S\,. \tag{21}$$

By replacing in (21) the equality

$$-E'(u_\epsilon)\frac{\partial^2 u_\epsilon}{\partial x^2} = E''(u_\epsilon)\left(\frac{\partial u_\epsilon}{\partial x}\right)^2 - \frac{\partial^2}{\partial x^2}[E(u_\epsilon)]$$

gives

$$0 = \frac{\partial}{\partial t}[E(u_\epsilon)] + \frac{\partial}{\partial x}[F(u_\epsilon)] - \epsilon\frac{\partial^2}{\partial x^2}[E(u_\epsilon)] + \epsilon E''(u_\epsilon)\left(\frac{\partial u_\epsilon}{\partial x}\right)^2$$

$$\implies \quad 0 \geq \frac{\partial}{\partial t}[E(u_\epsilon)] + \frac{\partial}{\partial x}[F(u_\epsilon)] - \epsilon\frac{\partial^2}{\partial x^2}[E(u_\epsilon)]\,,$$

since by definition $E \in C^2$ is convex $\Rightarrow E'' \geq 0$ in $\mathbb{R}$. □

Kružkov entropies

In the proof of the Kružkov uniqueness theorem, a very important role is played by the following specific class of entropy-flux couples.

Definition 15

The ∞^1 functions $E(u) = |u - k|$ $(k \in \mathbb{R})$ are called *Kružkov entropies*, with associated entropy flux

$$F(u) = [f(u) - f(k)]\, sgn\,(u - k) \qquad (k \in \mathbb{R})\,;$$

here

$$sgn\, r := \begin{cases} 1 & \text{se } r > 0 \\ 0 & \text{se } r = 0 \\ -1 & \text{se } r < 0\,. \end{cases}$$

Observe that $E, F \in C(\mathbb{R})$ and $F' = f'E'$ *a.e.* in $\mathbb{R}$.

The following proposition proves that, to establish whether a weak solution is entropic, *it is enough to check the entropy inequality for the Kružkov entropies*.

Proposition 6

Let $u_0 \in L^\infty(\mathbb{R})$, $u \in L^\infty(S)$. *Equivalent statements:*

(a) *u is an entropy solution of the Cauchy problem* (13)-(14)*;*

(b) *for any* $\zeta \in C_0^1(\overline{S})$, $\zeta \geq 0$ *and for any* $k \in \mathbb{R}$ *there holds*

$$\iint_S \{|u-k|\zeta_t + [f(u)-f(k)]\, sgn\,(u-k)\zeta_x\}\; dxdt + \qquad (22)$$

$$+\int_{\mathbb{R}} |u_0 - k|\zeta(x,0)\, dx \geq 0\,.$$

It is interesting to consider *entropy piecewise solutions* (for instance, because of the importance of the Riemann problem).

Lemma 16

Let $u \in C^1(\overline{\Omega}_\pm)$*. Equivalent statements:*
(a) u is an entropy solution of equation (13) *in* Ω*;*

$$(b)\ \begin{cases} \text{(i) } u \text{ is a classical solution of equation (13) in } \Omega_\pm; \\ \text{(ii) on } \gamma_0 \text{ the following inequality holds:} \\ \quad (*)\ \ [|u-k|]\xi' \geq [[f(u)-f(k)]\, sgn\,(u-k)] \quad \text{for any } k \in \mathbb{R}\,. \end{cases}$$

From the above lemma we get the famous *Oleinik entropy condition:*

Theorem 17

Let $u \in C^1(\overline{\Omega}_\pm)$*. Equivalent statements:*
(a) u is an entropy solution of equation (13) *in* Ω*;*

$$(b)\ \begin{cases} (i)\ u \text{ is a classical solution of equation (13) in } \Omega_\pm\,; \\ (ii) \text{ on } \gamma_0 \text{ for any } \alpha \in [0,1] \text{ there holds}: \\ (O)\quad \{\alpha f(u_-) + (1-\alpha) f(u_+) - \\ \qquad - f(\alpha u_- + (1-\alpha)u_+)\}\, sgn\,(u_+ - u_-) \leq 0\,. \end{cases}$$

Let us summarize:

So far we have discussed:

- Cubic ϕ with Sobolev regularization [Novick-Cohen & Pego]
- Singular limit as $\varepsilon \to 0$ [Plotnikov]: A few general remarks
- Hyperbolic first order conservation laws: A reminder

To be discussed yet:

- Singular limit as $\varepsilon \to 0$:
  - Entropy conditions and interfaces [Evans & Portilheiro]
  - Existence and uniqueness of two phase solutions

Problem (C) revisited

Let us recall Theorem 2, which proved existence of Young measure-valued solutions of problem (C):

There exist $u, v, \lambda_0, \lambda_1, \lambda_2 \in L^\infty(Q_T)$, $v \in C((0,T]; H^1(\Omega))$ such that:
(a) $\sum_{i=0}^{2} \lambda_i = 1$, $\lambda_i \geq 0$, $\lambda_1 = 1$ if $v < A$, $\lambda_2 = 1$ if $v > B$, and

$$u = \sum_{i=0}^{2} \lambda_i s_i(v)\,;$$

(b) for any $\zeta \in C^1(\overline{Q}_T)$, $\zeta(\cdot, T) = 0$ in $\overline{\Omega}$

$$\iint_{Q_T} (u\zeta_t - \nabla v \cdot \nabla \zeta)\, dxdt + \int_\Omega u_0 \zeta(x,0)\, dx = 0\,; \tag{23}$$

(c) for any $\zeta, g \in C^1(\mathbb{R})$, $g' \geq 0$

$$\iint_{Q_T} \Big\{ \sum_{i=0}^{2} \lambda_i G(s_i(v))\zeta_t - g(v)\nabla v \cdot \nabla \zeta - g'(v)|\nabla v|^2 \zeta \Big\} dxdt \geq 0\,. \tag{24}$$

Two-phase solutions

Now we want to use the analogy with the theory of hyperbolic conservation laws to address *two-phase solutions* of problem (C). To some extent, these are the counterpart of the piecewise smooth solutions considered in the theory of hyperbolic conservation laws.

Let us recall some notations and state again for convenience the relevant theorem.
We assume that

$$(H) \qquad \lambda_0 = 0 \;\; a.e. \text{ in } Q_T, \quad \lambda_i = 1 \; a.e. \text{ in } V_i \qquad (i = 1, 2) \;,$$

where

$$\overline{Q}_T = \overline{V}_1 \cup \overline{V}_2\,, \quad V_1 \cap V_2 = \emptyset$$

and

$$\gamma := \overline{V}_1 \cap \overline{V}_2$$

is a smooth surface.

Let $n = 1$ for simplicity. Then

$$\gamma = \{(\xi(t), t) \mid t \in [0, T]\}, \quad \xi \in C^1(0, T),$$

and the solutions of problem (C) given by Theorem 2 satisfy the following:

$$u = s_i(v) \ \ a.e.\ in\ V_i \quad (i = 1, 2);$$

$$\iint_{Q_T} (u\zeta_t - v_x\zeta_x)\, dxdt = \sum_{i=1}^{2} \iint_{V_i} \Big\{ s_i(v)\zeta_t - v_x\zeta_x \Big\} dxdt = 0 \qquad (25)$$

for any $\zeta \in C_0^\infty(Q_T)$;

$$\iint_{Q_T} \Big\{ G(u)\zeta_t - g(v)v_x\zeta_x - g'(v)|v_x|^2\zeta \Big\} dxdt =$$

$$= \sum_{i=1}^{2} \iint_{V_i} \Big\{ G(s_i(v))\zeta_t - g(v)v_x\zeta_x - g'(v)|v_x|^2\zeta \Big\} dxdt \geq 0 \qquad (26)$$

for any $g \in C^1(\mathbb{R})$, $g' \geq 0$, $\zeta \in C_0^\infty(Q_T)$, $\zeta \geq 0$.

Theorem 18

Let $u, v, \lambda_0, \lambda_1, \lambda_2$ be as in Theorem 2, with $\lambda_0, \lambda_1, \lambda_2$ satisfying assumption (H). Then u, v have the following properties:
(i) *u, v are classical solutions of*

$$u_t = v_{xx} \quad \text{in } V_i \qquad (i = 1, 2). \tag{27}$$

Besides, v is continuous in Q_T;
(ii) *there holds the* Rankine-Hugoniot condition:

$$\xi' = -\frac{[v_x]}{[u]} \quad a.e. \;\; on \;\gamma\,; \tag{28}$$

(iii) *there holds the* entropy condition:

$$\xi' \, [G(u)] \geq -g(v)[v_x] \quad a.e. \;\; on \;\gamma\,. \tag{29}$$

Here $[h] := h^+(\xi(t), t) - h^-(\xi(t), t)$ denotes the jump across γ of any piecewise continuous function h.

Hint of the proof of Theorem 18. We use the analogy with hyperbolic conservation laws. By definition, weak solutions of the Cauchy problem:

$$\begin{cases} u_t + [f(u)]_x = 0 \ \text{ in } \mathbb{R} \times (0, T] =: S_T \\ u = u_0 \ \ \text{ in } \mathbb{R} \times \{0\} \end{cases} \tag{30}$$

satisfy for any $\zeta \in C_0^\infty(S_T)$ the equality:

$$\iint_{S_T} \Big\{ u\zeta_t + f(u)\zeta_x \Big\} dxdt = 0 \,, \tag{31}$$

which corresponds to equality (25). Moreover, entropy solutions to (30) satisfy by definition the inequality

$$\iint_S \{E(u)\zeta_t + F(u)\zeta_x\} \ dxdt \geq 0 \tag{32}$$

for any couple entropy-flux (E, F) and any $\zeta \in C_0^\infty(S_T)$, $\zeta \geq 0 \ \Rightarrow$ counterpart of inequality (26).

If $\overline{S}_T = \overline{V}_1 \cup \overline{V}_2$, $V_1 \cap V_2 = \emptyset$ and $\gamma := \overline{V}_1 \cap \overline{V}_2$ as above, from equality (25) we obtain (28), as from (31) we obtained the Rankine-Hugoniot condition (Theorem 7):

$$\xi' = \frac{[f(u)]}{[u]} \quad \text{a.e. on } \gamma \tag{33}$$

(see Theorem 7). Similarly, from the inequality

$$\iint_{Q_T} \{G(u)\zeta_t - g(v)v_x\zeta_x\} dxdt \geq 0$$

(which follows from (26) since $g'(v)| \geq 0$) we obtain (29), as from (32) we obtained the entropy condition (see Lemma 16):

$$\xi' \, [E(u)] \geq [F(u)] \quad \text{a.e. on } \gamma \, . \tag{34}$$

It can be proved that $v_x(\cdot, \bar{t}) \in BV_{loc}(V_i \cap \{t = \bar{t}\})$ for a.e. $\bar{t} \in (0, T)$, thus the traces $v_x^{\pm}$ at γ are well defined. The continuity of v follows by Sobolev embedding, since $n = 1$.

Admissible phase changes and hysteresis

For *piecewise C^1 solutions* of the hyperbolic problem (30), different choices of the couple (E, F) in (34), where E is any Kružkov entropy and F the associated entropy flux, give the *Oleinik entropy condition*. This can be regarded as an *admissibility condition* for such a solution to be entropic.

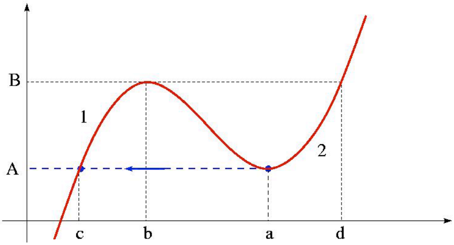
In the same way, from (29) by different choices of g we obtain *admissibility conditions* for the curve γ $\Rightarrow$ *admissible directions of propagation* of γ $\Rightarrow$ *admissible phase changes* $\Rightarrow$ *hysteresis effects* for solutions of problem (C). In fact, the following holds:

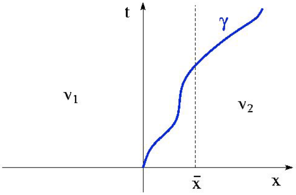
Theorem 19

Let u, v be as in Theorem 18. Then there holds:

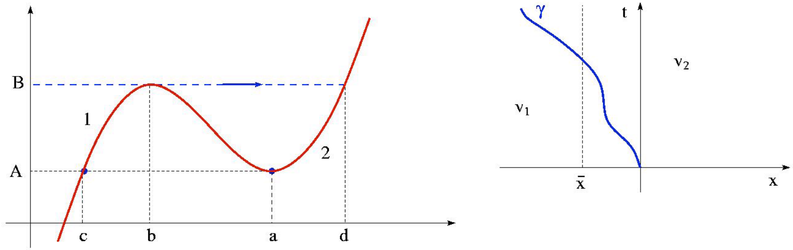
$$\begin{cases} (a) \;\; \xi' \geq 0 \text{ if } v = A\,; \\ (b) \;\; \xi' \leq 0 \text{ if } v = B\,; \\ (c) \;\; \xi' = 0 \text{ if } v \neq A,\; v \neq B\,. \end{cases} \tag{35}$$

Theorem 19, case (*a*)

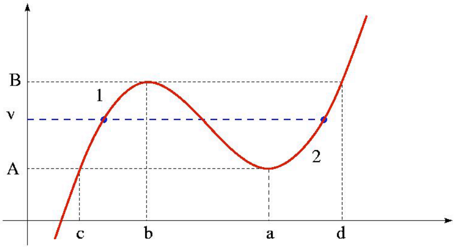


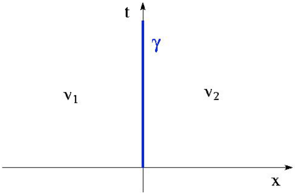


Theorem 19, case (*b*)



Theorem 19, case (c)





Hint of the proof of Theorem 19. Choose

$$g(t) = g_k(t) := sgn(t-k) \ \Rightarrow$$

$$\Rightarrow \ G_k(u) = \int_0^u sgn(\phi(t)-k)dt \quad (k,t \in \mathbb{R})$$

(analogous of the *Kruzkov entropies* for problem (30)). From (28)-(29) we obtain:

$$\xi' \left\{ [G(u)] - g(v)[u] \right\} \geq 0 \quad \text{a.e. on } \gamma$$

for any nondecreasing g, thus by the above choice for any $k \in \mathbb{R}$

$$\xi' \int_{u^-}^{u^+} \left\{ sgn(\phi(t)-k) - sgn(\phi(u)-k) \right\} dt \geq 0$$

When $v = \phi(u) = A$ choose $k \in (A,B)$. Then

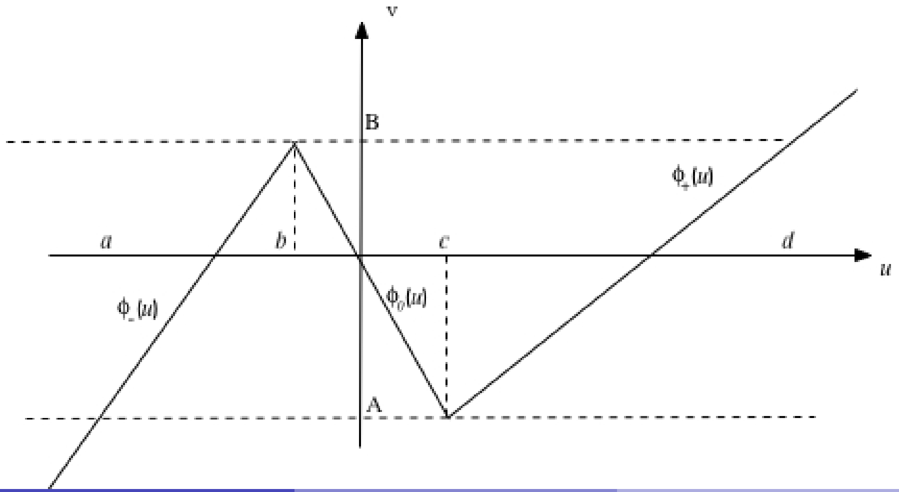
$$\phi(u) - k = A - k < 0 \ \Rightarrow \ sgn(\phi(u)-k) = -1 \ \Rightarrow$$

$$\Rightarrow \ sgn(\phi(t)-k) - sgn(\phi(u)-k) \geq 0 \ ,$$

whence $\xi' \geq 0$ by the above inequality. This proves claim (a). To prove (b) we also choose $k \in (A,B)$, while for (c) we choose first $k \in (v,B)$, then $k \in (A,v)$; hence the conclusion.

The above results describe the behaviour of two-phase entropy solutions, *if they exist at all*. However, they do not say anything about their actual *existence* **and** *uniqueness*. **Relying on them, we can prove the theorem below.**

To avoid technicalities, we assume ϕ to be *piecewise linear:*



Let us make the following assumption on the initial data:

$$\begin{cases} (i) & u_0(\mathbb{R}_-) \subseteq (-\infty, a],\ u_0(\mathbb{R}_+) \subseteq [b, \infty)\,; \\ (ii) & u_0 \in H^{2,\infty}(\mathbb{R}_-) \cap H^{2,\infty}(\mathbb{R}_+), \\ & \lim_{x \to \pm\infty} u_0'(x) = 0\,; \\ (iii) & \lim_{\eta \to 0^+} \phi(u_0)(-\eta) = \lim_{\eta \to 0^+} \phi(u_0)(\eta)\,. \end{cases} \qquad (A)$$

Theorem 7 (Mascia, Terracina & T.)

Let ϕ be piecewise linear, and assumption (A) hold. Then there exists *a two-phase solution local in time. Moreover, if the functions $\phi(u_0) - A$, $\phi(u_0) - B$ change sign at most a finite number of times in any compact subset of $\mathbb{R}$, the two-phase solution is* unique*.*

Idea of the proof. The entropy conditions suggest two *auxiliary problems*, called the *moving boundary problem* and the *steady boundary problem*. We first solve them, then "glue together" their solutions properly.

References

- L. C. Evans, *A survey of entropy methods for partial differential equations*, Bull. Amer. Math. Soc. **41** (2004), 409-438.
- L. C. Evans & M. Portilheiro, *Irreversibility and hysteresis for a forward-backward diffusion equation*, Math. Mod. Meth. Appl. Sci. **14** (2004), 1599-1620.
- B. H. Gilding & A. Tesei, *The Riemann problem for a forward–backward parabolic equation*, Physica D **239** (2010), 291-311.
- M. Gurtin, *Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance*, Physica D **92** (1996), 178-192.
- P. Lafitte & C. Mascia, *Numerical exploration of a forward–backward diffusion equation*, Math. Mod. Meth. Appl. Sci. **22** (2012) (doi: 10.1142/S0218202512500042).

References

C. Mascia, A. Terracina & A. Tesei, *Evolution of stable phases in forward-backward parabolic equations*, in: *Asymptotic Analysis and Singularities* (edited by H. Kozono, T. Ogawa, K. Tanaka, Y. Tsutsumi & E. Yanagida), pp. 451-478, Advanced Studies in Pure Mathematics **47-2** (Math. Soc. Japan, 2007).

C. Mascia, A. Terracina & A. Tesei, *Two-phase entropy solutions of a forward-backward parabolic equation,* Arch. Rational Mech. Anal. **194** (2009), 887-925.

A. Novick-Cohen & R. L. Pego, *Stable patterns in a viscous diffusion equation*, Trans. Amer. Math. Soc. **324** (1991), 331-351.

P. I. Plotnikov, *Passing to the limit with respect to viscosity in an equation with variable parabolicity direction*, Diff. Equ. **30** (1994), 614-622.

References

- P. I. Plotnikov, *Equations with alternating direction of parabolicity and the hysteresis effect*, Russian Acad. Sci. Dokl. Math. **47** (1993), 604-608.
- P. I. Plotnikov, *Forward-backward parabolic equations and hysteresis*, J. Math. Sci. **93** (1999), 747-766.
- D. Serre, *Systems of Conservation Laws, Vol. 1: Hyperbolicity, Entropies, Shock Waves* Cambridge University Press, Cambridge (1999).
- F. Smarrazzo & A. Terracina, *Sobolev approximation for two-phase solutions of forward-backward parabolic problems*, Discrete Contin. Dyn. Syst. **33** (2013), 1657-1697.
- M. Valadier, *A course on Young measures*, Rend. Ist. Mat. Univ. Trieste **26** (1994), Suppl., pp. 349-394.