Two topics in the theory of reaction-diffusion equations

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Forward-backward parabolic equations, 3

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Plan of the lecture

- Perona-Malik ϕ with Sobolev regularization
- Perona-Malik ϕ with Barenblatt regularization:
  - Monotonicity and regularity properties of solutions
  - Formation of singularities

Problem: Radon measure-valued solutions of the equation

$$u_t = \Delta[\phi(u)] + \epsilon\Delta[\psi(u)]_t \,. \qquad (1)$$

Main features of the problem:

(a) $\phi : \mathbb{R} \to \mathbb{R}$ *is nonmonotonic, odd* and

$$\phi(u) \to 0 \quad \text{as} \quad u \to \pm\infty \,;$$

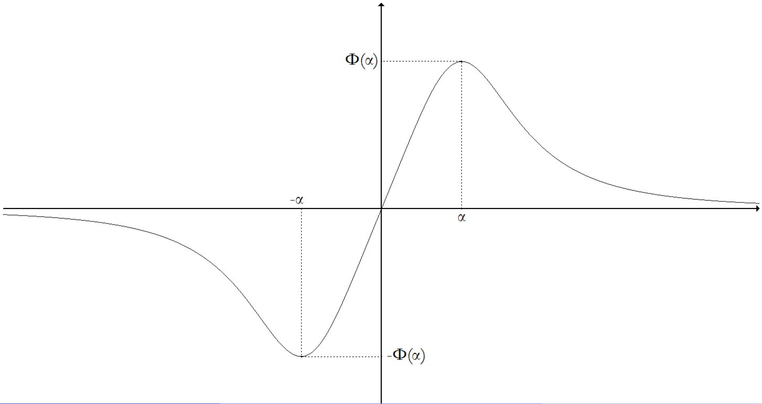
(b) $\psi : \mathbb{R} \to \mathbb{R}$ *is increasing, odd* and

$$\psi(u) \to \pm\gamma \in \mathbb{R} \quad \text{as} \quad u \to \pm\infty \qquad (\gamma > 0) \,.$$

The regularization used in (1) is called *Barenblatt regularization*. It is weaker than the Sobolev regularization, which formally corresponds to the choice $\psi(u) = u$, since $\psi'(u) \to 0$ as $u \to \pm\infty$. Therefore, the Barenblatt regularization is "ineffective" for large values of u, and the problem is *degenerate pseudoparabolic*.

In image processing, population dynamics, oceanography: ϕ "of Perona-Malik type"

$$(A) \qquad \begin{cases} \phi \in C^2(\mathbb{R}) \cap L^1(\mathbb{R}),\ \phi \text{ odd};\ \phi(u) > 0 \ \text{ if } u > 0\,, \\ \phi(0) = 0,\ \phi(u) \to 0 \text{ if } u \to \pm\infty\,, \\ \phi'(u) > 0 \ \text{ if } 0 \le u < \alpha,\ \phi'(u) < 0 \ \text{ if } u > \alpha\,. \end{cases}$$



Motivating the choice of the regularization [Barenblatt]:

By taking into account *time delay effects*, the "right model equation" is

$$u_t(x,t) = \Delta\left[u(x,t)\, k_0(u(x,t-\epsilon))\right] \quad \text{with } k_0(u) := \frac{\phi(u)}{u}\,,$$

where the small parameter ϵ represents a *relaxation time* [BBDPU]. A *formal development* with respect to ϵ gives equation (1):

$$u_t = \Delta[\phi(u)] + \epsilon\Delta[\psi(u)]_t\,,$$

$$\text{where} \quad \psi(u) := -\phi(u) + \int_0^u k_0(p)\, dp\,.$$

A typical choice is

$$\phi(u) = \frac{u}{1+u^2} \quad \Rightarrow \quad \psi(u) = -\frac{u}{1+u^2} + \mathrm{arctg}\, u \to \pm\frac{\pi}{2} \quad \text{as } u \to \pm\infty\,.$$

The regularization is degenerate pseudoparabolic, since $\psi'(u) \to 0$ as $u \to \pm\infty$.

Let us first use the Sobolev regularization:

Consider the problem

$$(C)\qquad \begin{cases} u_t = \Delta[\phi(u)] & \text{in } \Omega \times (0,T) =: Q \\ \dfrac{\partial \phi(u)}{\partial n} = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 \in L^1(\Omega) & \text{in } \Omega \times \{0\} \end{cases}$$

with ϕ of Perona-Malik type.

Consider the Sobolev regularized problem

$$(C_\epsilon)\qquad \begin{cases} u_t = \Delta v & \text{in } Q \\ \dfrac{\partial v}{\partial n} = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 \in L^1(\Omega) & \text{in } \Omega \times \{0\}\,, \end{cases}$$

where the *chemical potential* v is defined as

$$v := \phi(u) + \epsilon u_t\,.$$

Remark: The situation is the same as in [Novick-Cohen & Pego] for cubic ϕ.

Definition 1

Let $u_0 \in L^\infty(\Omega)$. By a *solution* to problem (C_ϵ) in Q_T we mean any couple $u \in C^1([0,T]; L^\infty(\Omega))$, $v \in C([0,T]; L^\infty(\Omega) \cap W^{2,p}(\Omega))$ with $p > n$, $\Delta v \in C([0,T]; L^\infty(\Omega))$ which satisfies (C_ϵ) in the strong sense.

Existence and uniqueness:

Theorem 1

For any $\epsilon > 0$ there exists a unique solution (u_ϵ, v_ϵ) of problem (C_ϵ).

In particular, we have the weak equality:

$$\iint_Q (u_\epsilon \zeta_t - \nabla v_\epsilon \cdot \nabla \zeta)\, dxdt + \int_\Omega u_0 \zeta(x,0)\, dx = 0 \qquad (2)$$

for any $\zeta \in C^1([0,T]; C^1(\overline{\Omega}))$, $\zeta(\cdot, T) = 0$ in Ω.

The main inequality of the Sobolev regularization holds true:

For any $g \in C^1(\mathbb{R})$, $g' \geq 0$ set

$$G(z) := \int_0^z g(\phi(s))ds + c\,. \qquad (3)$$

For any solution (u_ϵ, v_ϵ) of (C_ϵ),

$$\begin{aligned}[G(u_\epsilon)]_t &= g(\phi(u_\epsilon))u_{\epsilon t} = g(\phi(u_\epsilon))\Delta v_\epsilon = \\ &= div\Big[g(v_\epsilon)\nabla v_\epsilon\Big] - g'(v_\epsilon)|\nabla v_\epsilon|^2 + \\ &+ \underbrace{\Big[g(\phi(u_\epsilon)) - g(v_\epsilon)\Big]\frac{v_\epsilon - \phi(u_\epsilon)}{\epsilon}}_{\leq 0}\,. \end{aligned} \qquad (4)$$

Therefore,

$$\frac{d}{dt}\int_\Omega G(u_\epsilon(x,t))\,dx \leq 0\,. \qquad (5)$$

As for cubic ϕ, from inequality (5) we get:

Proposition 2

Assume that

$$\phi(u_1) \leq \phi(u) \leq \phi(u_2) \quad \textit{for any} \quad u \in [u_1, u_2]; \tag{6}$$

moreover, let $u_0(x) \in [u_1, u_2]$ for any $x \in \Omega$. Then $u_\epsilon(x,t) \in [u_1, u_2]$ for any $(x,t) \in Q$.

However, for ϕ of Perona-Malik the only *bounded* invariant regions are intervals $[u_1, u_2] \subseteq [-\alpha, \alpha]$, a case of little interest.

A more interesting consequence of (5) is the *conservation of positivity:*

Proposition 3

Let ϕ satisfy (A), and let $u_0(x) \geq 0$ for a.e. $x \in \Omega$. Then $u_\epsilon(x,t) \geq 0$ for a.e. $(x,t) \in Q$.

The proof is the same as for Proposition 2.

A priori estimates...

By Proposition 3, if $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, the *conservation of mass* gives an estimate of $\{u_\epsilon\}$ in $L^\infty((0,T);L^1(\Omega))$, *uniform with respect to* ϵ:

$$\overbrace{\|u_\epsilon(\cdot,t)\|_{L^1(\Omega)} =}^{\textit{Proposition 3}} \underbrace{\int_\Omega u_\epsilon(x,t)\,dx = \int_\Omega u_0(x)\,dx}_{\text{mass conservation}} \overbrace{= \|u_0\|_{L^1(\Omega)}}^{u_0\geq 0} .$$

Since $L^1(Q) \subset \mathcal{M}(Q) \equiv \{\text{finite Radon measures on } Q\}$, by weak* compactness in $\mathcal{M}(Q)$ there exist a sequence $\{u_{\epsilon_j}\}$ and a Radon measure $u \in \mathcal{M}^+(Q)$ such that

$$\iint_Q u_{\epsilon_j}\, f\, dxdt \ \to < u, f >_Q \ \text{ for any } f \in C_c(Q)\,;$$

here $< \cdot, \cdot >$: $C_c(Q) \times \mathcal{M}(Q) \mapsto \mathbb{R}$ denotes the duality map.

More precise information is given by the Chacón Biting Lemma.

Lemma 4 (Chacón)

There exist a sequence $\{u_{\epsilon_j}\}$ and a sequence $A_{j+1} \subseteq A_j \subseteq Q$, with Lebesgue measure $|A_j| \to 0$ as $j \to \infty$, such that:

- *the sequence $\left\{u_{\epsilon_j}\chi_{Q\setminus A_j}\right\}$ is weakly compact in $L^1(Q)$, thus there exists $u_r \in L^1(Q)$ such that*

$$\iint_Q u_{\epsilon_j}\chi_{Q\setminus A_j}\, f\, dxdt \;\to \iint_Q u_r\, f\, dxdt \;\text{ for any } f \in L^\infty(Q)\,;$$

- *there exists a measure $u_s \in \mathcal{M}^+(Q)$ such that*

$$\iint_Q u_{\epsilon_j}\chi_{A_j}\, f\, dxdt \;\to < u_s, f >_Q \;\text{ for any } f \in C_c(Q)\,.$$

Then from equality (2) we get $\forall \zeta \in C^1([0,T]; C^1_c(\Omega))$, $\zeta(\cdot, T) = 0$ in Ω

$$\iint_Q (u_r\zeta_t - \nabla v \cdot \nabla\zeta)\, dxdt + \int_\Omega u_0\zeta(x,0)\, dx = - < u_s, \zeta_t >_Q\,.$$

... and their consequences

The above remarks suggest the following:

If ϕ is of Perona-Malik type, solutions of problem (C) can take values in $\mathcal{M}^+(\Omega)$ for positive times *even if the initial data function u_0 belongs to $L^1(\Omega)$*.

This corresponds to

"deregularizing properties" of problem (C)

$\leftrightsquigarrow$ *spontaneous appearance of singularities for positive times*.

Remark: In the above discussion use has been made of the *Sobolev regularization*, which is *stronger than the Barenblatt regularization* used in (1). This is why *Radon measures only appear as $\epsilon \to 0$.*

Actually, Radon measures appear *even for $\epsilon > 0$* for the regularized problem with *the Barenblatt regularization*.

In fact, it was shown in [BBDPU] that solutions of the oceanography model

$$\begin{cases} z_t = [\phi(z_x)]_x + \epsilon[\psi(z_x)]_{tx} & \text{in } (0,1)\times(0,T) \\ \phi(z_x) + \epsilon[\psi(z_x)]_t = 0 & \text{in } \{0,1\}\times(0,T) \\ z = z_0 \in BV((0,1)) & \text{in } (0,1)\times\{0\} \end{cases}$$

with $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi'(s) > 0$, $\psi(s) \to \pm\gamma$ as $s \to \pm\infty$ *may become discontinuous (with respect to x) for* $t > 0$, even if $z_0 \in C^1([0,1])$.

In terms of $u := z_x$ this means that *Dirac masses may appear for* $t > 0$ in the solution of the problem

$$\begin{cases} u_t = [\phi(u)]_{xx} + \epsilon[\psi(u)]_{txx} & \text{in } (0,1)\times(0,T) \\ \phi(u) + \epsilon[\psi(u)]_t = 0 & \text{in } \{0,1\}\times(0,T) \\ u = u_0 \in \mathcal{M}^+(\Omega) & \text{in } (0,1)\times\{0\}, \end{cases}$$

even if $u_0 \in C([0,1])$.

Joint work with M. Bertsch & F. Smarrazzo

Problem: Existence and qualitative properties of Radon measure-valued solutions to the parabolic problem

$$(P) \qquad \begin{cases} u_t = \Delta v & \text{in } Q \\ v = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 \in \mathcal{M}^+(\Omega) & \text{in } \Omega \times \{0\}, \end{cases}$$

where now the *chemical potential* v is defined by the Barenblatt regularization as

$$v := \phi(u) + \epsilon[\psi(u)]_t\,,$$

and $\mathcal{M}^+(\Omega) \equiv \{\text{positive finite Radon measures on } \Omega\}$.

Question: How to make sense of $\phi(u(\cdot,t))$ and $\psi(u(\cdot,t))$, if $u(\cdot,t) \in \mathcal{M}^+(\Omega)$?

General idea [BBDU]: We use the Lebesgue decomposition $u = u_r + u_s$, but the right-hand side of $u_t = \Delta v$ *only depends on the function* $u_r \in L^1(Q)$:

$$u_t = (u_r + u_s)_t = \Delta\left[\phi(u_r) + \epsilon[\psi(u_r)]_t\right].$$

A side remark: The porous medium equation

The same definition could be used for the porous medium equation

$$u_t = \Delta u^m \qquad (m > 0)\,. \tag{7}$$

- [Brezis & Friedman, 1983] The Cauchy-Dirichlet problem for (7)
  - has a solution in $L^1(Q)$ for any $u_0 \in \mathcal{M}(\Omega)$ if
  $$m > m_c := \frac{(n-2)_+}{n} \qquad (n \geq 3)\,;$$
  - does not have solutions in $L^1(Q)$, if $u_0 = \delta$ is the Dirac mass and $0 < m \leq m_c$.
- [Pierre, 1985] The Cauchy problem in $\mathbb{R}^N$ for equation (7) with $m \leq m_c$ has a solution in $L^1(\mathbb{R}^N)$ iff $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$ is a *measure diffuse with respect to a capacity* depending on m:
  $$C_{2,\frac{1}{1-m}}(K) = 0 \quad \Rightarrow \quad u_0(K) = 0\,.$$

Remark: Even if u_0 is a Radon measure, only *function-valued solutions* were considered.

In the case of the porous medium equation, the solution for $t > 0$ is more regular than $u_0 \in \mathcal{M}^+(\Omega)$ "if m is large", namely, if ϕ increases fast enough at infinity.

In the Perona-Malik case, ϕ decreases at infinity; moreover, the Barenblatt regularization is not effective at infinity since $\psi'(u) \to 0$ as $u \to \infty$. Therefore, we expect the solution for $t > 0$ to be less regular than u_0. This expectation is confirmed by the following

Theorem 5 (Monotonicity of the singular part)

Let u be a solution of problem (P), and let u_s denote its singular part in the Lebesgue decomposition. Then for any $\rho \in C_c(\Omega)$, $\rho \geq 0$

$$\langle u_s(\cdot, t_1), \rho\rangle_\Omega \leq \langle u_s(\cdot, t_2), \rho\rangle_\Omega \quad \text{for a.e. } 0 \leq t_1 \leq t_2 \leq T\,.$$

Remark: Existence of suitably defined solutions of (P) is proven by an approximation procedure: (i) we approximate ψ and u_0 in problem (P) by suitable sequences ψ_n, $\psi_n' \geq \frac{1}{n}$, and $\{u_{0n}\} \subseteq C_c^\infty(\Omega)$, $u_{0n} \geq 0$; (ii) we study the limiting points of the sequence of solutions of the corresponding *approximating problems*.

Formation of singularities:

Spatial singularities can appear spontaneously for positive times:

Consider for any $t \in [0, T]$ the "good" set

$$\mathcal{B}(t) := \{x \in \overline{\Omega} \mid \exists\, I_\delta(x) \text{ s. t. } u_s(\cdot, t) = 0 \text{ in } I_\delta(x), u_r(\cdot, t) \in L^\infty(I_\delta(x))\} .$$

It is proven that:

- for every $0 \le t_1 \le t_2 \le T$, $\mathcal{B}(t_2) \subseteq \mathcal{B}(t_1)$ (*i.e.*, $|\mathcal{B}(\cdot)|$ is decreasing) ;
- for every $t \in [0, T]$ there holds $v(\cdot, t) = 0$ in $\overline{\Omega} \setminus \mathcal{B}(t)$;
- in $\mathcal{D}'(\mathcal{B}(t) \cap \Omega)$ we have the *elliptic equation*

$$-\epsilon v_{xx}(\cdot, t) + \frac{v(\cdot, t)}{\psi'(u_r)(\cdot, t)} = \frac{\phi(u_r)(\cdot, t)}{\psi'(u_r)(\cdot, t)} \quad \Leftrightarrow \quad \begin{cases} u_t = v_{xx} \\ v = \phi(u_r) + \epsilon[\psi(u_r)]_t \, . \end{cases}$$

Since $v(\cdot, t) = 0$ in $\overline{\Omega} \setminus \mathcal{B}(t)$, by the *strong maximum principle* and the *Hopf Lemma* $v_x(\cdot, t)$ has a jump discontinuity at any $x_0 \in \partial\mathcal{B}(t)$ where

$$\text{the map } x \to \frac{|x - x_0|}{\psi'(u_r)(x, t)} \text{ belongs to } L^1_{loc}(\Omega) \, .$$

Hence *a Dirac mass for v_{xx} can appear, if v vanishes.*

How to address in general the formation of singularities?

- *Heuristic idea:* To make use of *time reversal,*

$$u(x,t) \Rightarrow z(x,t) := u(x,1-t)\,,$$

$$\phi(u) \Rightarrow \chi(z) := \phi(\alpha) - \phi(z)\,.$$

Expected correspondence:

formation of singularities for problem (P)

$\leftrightsquigarrow$ *disappearance of singularities of the "backward problem"*.

Problem: To study the "backward problem" corresponding to (P), in particular its *regularizing properties* .

Backward problem:

Let $T \geq 1$, $u_0 \in \mathcal{M}^+(\Omega)$, and let (u, v) be the corresponding solution of problem (P). Since $u \in C([0,T];\mathcal{M}^+(\Omega))$, the function

$$z : [0,1] \to \mathcal{M}^+(\Omega)\,, \quad z(\cdot,t) := u(\cdot, 1-t)$$

is defined, and $z \in C([0,T];\mathcal{M}^+(\Omega))$. Plainly, z satisfies in a suitable sense the following *backward problem:*

$$(B) \qquad \begin{cases} z_t = [\chi(z)]_{xx} + \epsilon[\psi(z)]_{txx} & \text{in } \Omega \times (0,1) =: Q_1 \\ \chi(z) + \epsilon[\psi(z)]_t = \phi(\alpha) & \text{in } \partial\Omega \times (0,1) \\ z = p_0 := u(\cdot, 1) & \text{in } \Omega \times \{0\}\,, \end{cases}$$

where

$$\chi(s) := -\phi(s) + \phi(\alpha) \qquad (s \in [0,\infty))\,. \tag{8}$$

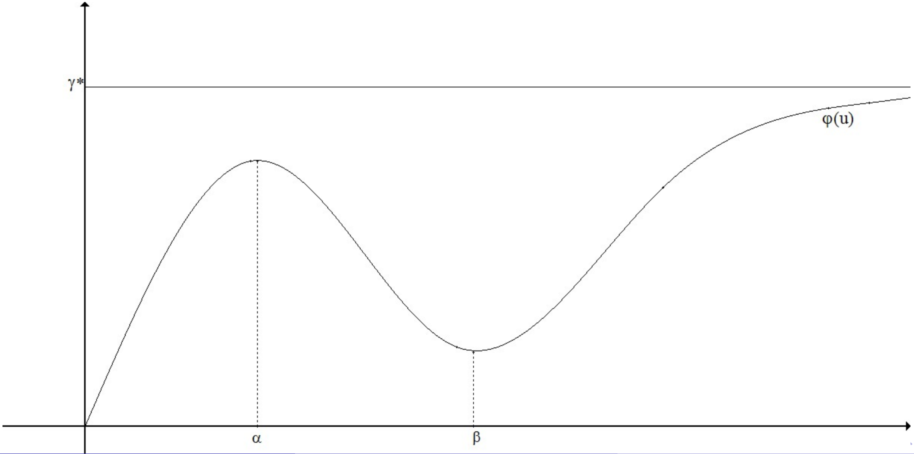
The qualitative properties of (B) are "dual" of those of (P). In particular, *the singular part of the solution is nonincreasing* (see Theorem 5):

$$\langle u_s(\cdot,t_1),\rho\rangle_\Omega \geq \langle u_s(\cdot,t_2),\rho\rangle_\Omega \quad \textit{for a.e. } 0 \leq t_1 \leq t_2 \leq T\,.$$

In view of (A), the function χ satisfies the following assumption:

$$\begin{cases} (i) \quad \chi \in C^{\infty}([0,\infty)),\ \chi(0) = \phi(\alpha),\ \chi(\alpha) = 0,\ \chi(u) \to \phi(\alpha) \ \text{as}\ u \to \infty; \\ (ii)\ \chi' < 0 \ \text{in}\ [0,\alpha),\ \chi' > 0 \ \text{in}\ (\alpha,\infty),\ \chi''(\alpha) \neq 0\,; \\ (iii) \quad \chi^{(j)} \in L^{\infty}(0,\infty) \ \text{for any}\ j \in \mathbb{N}\,. \end{cases}$$

For our purposes, we can make use of the following function φ:



Formation of singularities: Main result

Theorem 2

Let $\Omega = (-3,3)$*, and let assumption* (A) *be satisfied. Let* $p_0 \in \mathcal{M}^+(\Omega)$ *satisfy the following properties:*
(i) p_0 *is even;*
(ii) $p_{0r} \geq \alpha$ *a.e. in* Ω*;*
(iii) p_{0s} *is concentrated on the set* $(-2,-1) \cup (1,2)$*;*
(iv) *there holds* $\{p_{0r} < \infty\} \subseteq \mathcal{B}_{p_0}$*, where*

$$\mathcal{B}_{p_0} := \{x \in \overline{\Omega} \mid \exists\, I_\delta(x) \text{ such that } p_{0s} = 0 \text{ in } I_\delta(x) \text{ and } p_{0r} \in L^\infty(I_\delta(x))\}\,.$$

Then there exists $\xi_0 > 0$ *such that: If* $6\alpha \leq p_0(\Omega) \leq 6\alpha + \xi_0$ *, there exists* $u_0 \in L^\infty_{loc}(\Omega)$*,* $u_0 \geq 0$ *and a corresponding solution* (u,v) *of problem* (P) *with* $T \geq 1$ *such that*

$$u(\cdot,1) = p_0\,.$$

Examples

- Examples can be constructed where:
  1. p_{0s} is the sum of countably many Dirac masses:

$$p_{0s} = \sum_{n=1}^{\infty} b_n \left(\delta_{x_n+1} + \delta_{-(x_n+1)}\right) , \quad b_n \geq 0 \ \forall n \in \mathbb{N} , \quad \sum_{n=1}^{\infty} b_n < \infty ;$$

  2. p_{0s} is "Cantor-like", thus *singular continuous:*

$$p_{0s} = [\mu_C(\cdot - 1) + \mu_C(1 - \cdot)]\, \Omega ,$$

  where μ_C denotes the Cantor measure.

To summarize:

- very general phenomena of formation of singularities;
- connection with a new, challenging **regularity problem for measure-valued solutions**.

References

- G. I. Barenblatt, M. Bertsch, R. Dal Passo & M. Ughi, *A degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow,* SIAM J. Math. Anal. **24** (1993), 1414-1439.
- M. Bertsch, F. Smarrazzo & A. Tesei, *Pseudoparabolic regularization of forward-backward parabolic equations: Power-type nonlinearities,* J. Reine Ang. Math. (to appear).
- M. Bertsch, F. Smarrazzo & A. Tesei, *Pseudoparabolic regularization of forward-backward parabolic equations: A logarithmic nonlinearity,* Analysis & PDE 6-7 (2013), 1719 - 1754.
- M. Bertsch, F. Smarrazzo & A. Tesei, *Pseudoparabolic regularization of forward-backward parabolic equations: Bounded nonlinearities,* preprint (2015).

References

M. Giaquinta, G. Modica & J. Souček, *Cartesian Currents in the Calculus of Variations, Vol. I* (Springer, 1998).

F. Smarrazzo, *On a class of equations with variable parabolicity direction*, Discr. Contin. Dyn. Syst. **22** (2008), 729-758.

F. Smarrazzo & A. Tesei, *Degenerate regularization of forward-backward parabolic equations: The regularized problem*, Arch. Rational Mech. Anal. 204 (2012), 85-139.

F. Smarrazzo & A. Tesei, *Degenerate regularization of forward-backward parabolic equations: The vanishing viscosity limit*, Math. Ann. 355 (2013), 551-584.